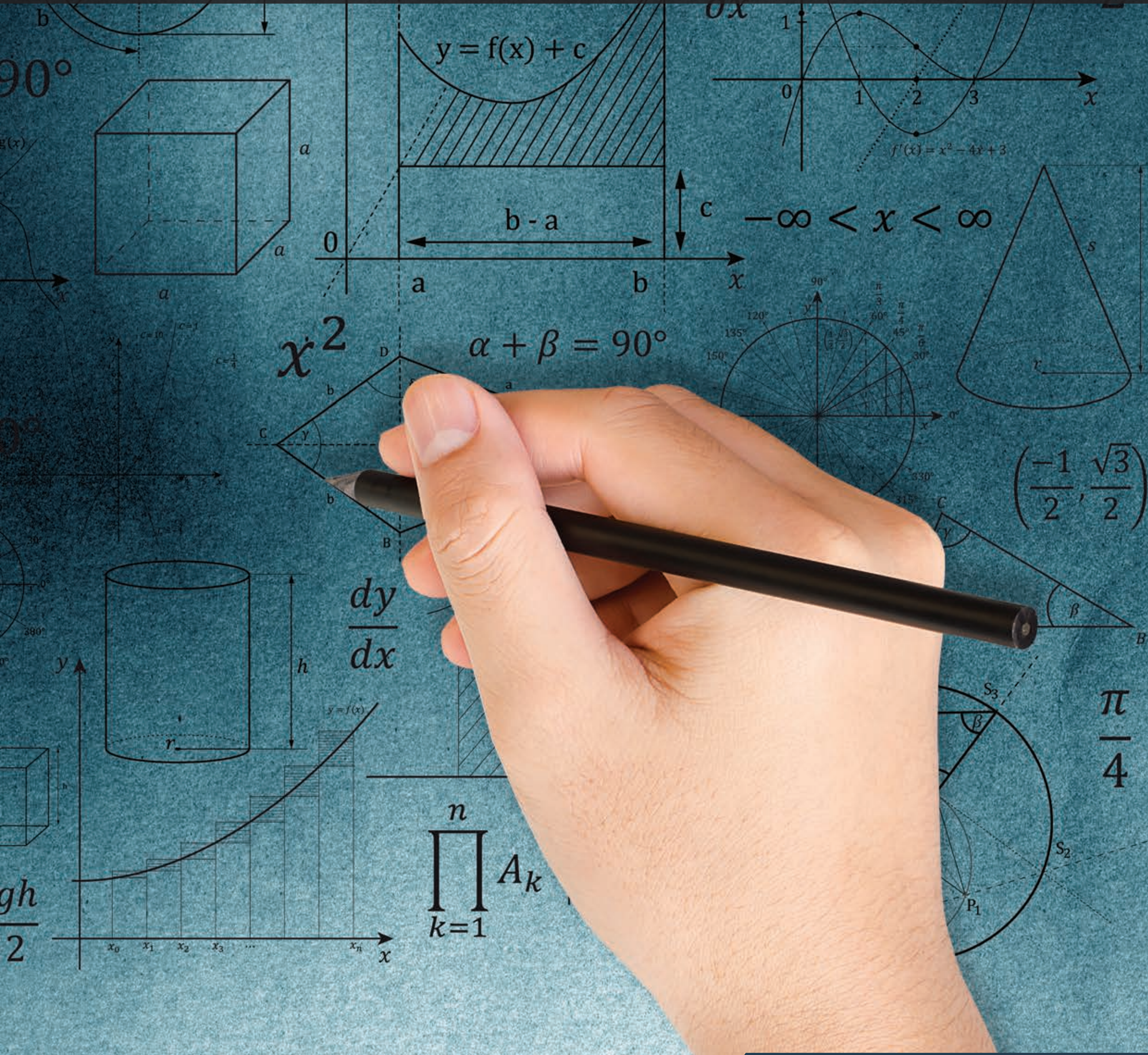
Geometric reasoning: Year 9

MATHEMATICS CONCEPTUAL NARRATIVE

Leading Learning: Making the Australian Curriculum work for us by bringing CONTENT and PROFICIENCIES together



www.acleadersresource.sa.edu.au

Government of South Australia
Department for Education and Child Development

Contents

Resource key

The ‘**AC**’ icon indicates the Australian Curriculum: Mathematics content description(s) addressed in that example.

The ‘**From tell to ask**’ icon indicates a statement that explains the transformation that is intended by using the task in that example.

More information about ‘Transforming Tasks’:
http://www.acleadersresource.sa.edu.au/index.php?page=into_the_classroom

Look out for the purple pedagogy boxes, that link back to the SA TfEL Framework.

The ‘**Bringing it to Life (BitL)**’ tool icon indicates the use of questions from the *Leading Learning: Making the Australian Curriculum Work for Us* resource.

Bringing it to Life (BitL) key questions are in bold orange text.

Sub-questions from the BitL tool are in green medium italics – these questions are for teachers to use directly with students.

More information about the ‘Bringing it to Life’ tool:
http://www.acleadersresource.sa.edu.au/index.php?page=bringing_it_to_life

Throughout this narrative—and summarised in **‘Geometric reasoning’ from Foundation to Year 10A** (see page 23)— we have colour coded the AC: Mathematics year level content descriptions to highlight the following curriculum aspects of working with geometric reasoning:

- Estimate/measure/compare/identify/classify using geometrical reasoning
- Investigate and generalise using geometrical reasoning
- Solve and prove using geometrical reasoning.

What the Australian Curriculum says about 'geometric reasoning'

Content descriptions

Strand | Measurement and geometry.

Sub-strand | Geometric reasoning.

Year 9 ◆ ◆ | ACMMG220
Students use the enlargement transformation to explain similarity and develop the conditions for triangles to be similar.

Year 9 ◆ | ACMMG221
Students solve problems using ratio and scale factors in similar figures.

Year level descriptions

Year 9 ◆ | Students apply ratio and scale factors to similar figures.

Year 9 ◆ | Students develop strategies in investigating similarity.

Achievement standards

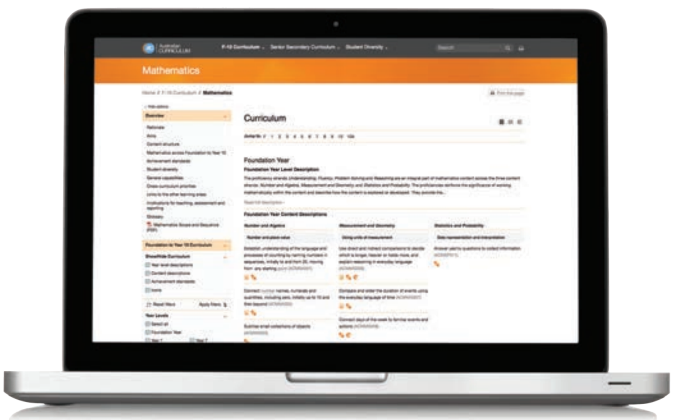
Year 9 ◆ | Students interpret ratio and scale factors in similar figures.

Year 9 ◆ | Students explain similarity of triangles.

Year 9 ◆ | Students recognise the connections between similarity and the trigonometric ratios.

Numeracy continuum

End of Year 10 ◆ ◆ | Students visualise, describe and analyse the way shapes and objects are combined and positioned in the environment for different purposes (Using spatial reasoning: Visualise 2D shapes and 3D objects).



Source: ACARA, Australian Curriculum: Mathematics, Version 8.1

Working with ‘geometric reasoning’

Important things to notice about this sub-strand of the Australian Curriculum: Mathematics and numeracy continuum

What we are building on and leading towards in Year 9 geometric reasoning

In Year 8 students use their observations of the properties of quadrilaterals to sort, classify and solve problems involving quadrilaterals. The term ‘congruence’ is introduced in Year 8 and students establish the (minimum) conditions necessary for congruence of triangles.

In Year 9 students continue to make observations about angles and side lengths of triangles and they use observations together with an understanding of enlargement and ratio to explain similarity. Students establish the conditions for similarity in triangles and they compare these observations to the conditions required for congruence of triangles.

In Year 10 students establish the difference between demonstration and proof. Students apply their understanding of congruence and angle properties to deduce properties of other geometric figures. They also use congruence and similarity to calculate unknown angles, even when there are multiple steps involved in reaching the solution.

In Year 10A students extend their geometric reasoning to prove and apply angle and chord properties of circles.

- **Notice that it is *only* in the numeracy continuum that we see an acknowledgement of the need for students to *visualise* shapes.** Visualisation is an important skill for young mathematicians to be developing especially when they are formulating geometric arguments. The numeracy continuum also states the need to *describe and apply* their understanding of the features and properties of 2D shapes and 3D objects. This could be easily overlooked when only reading the content strands and achievement standards.
- Although the AC Mathematics states that Year 9 students were expected to ***develop*** the conditions for similar triangles, it is not explicitly stated that students need to be able to communicate their reasoning using formal protocols. Year 9 students will be ***comparing the conditions required for similarity to those for congruence***. It is essential to revisit the ideas developed in Year 8 and explicitly model and scaffold the protocols of recording formal proofs.
- **Practice on its own is not enough.** Often we think students will get better at formal proof if we just give them enough practice, but we need to be more deliberate and account for it in our learning design. Students benefit from the opportunity to use some different strategies, provide feedback to a fictitious student’s attempt at proof, reconstruct a ‘jigsaw’ of a formal written solution, do an audio tape of a spoken proof etc.
- **At this stage of development, students often have difficulty distinguishing between:**
  - **the information you can use as truth**
  - **things that look like they are true**
  - **what you are trying to prove.**

  As teachers we should support students to develop a habit of clarifying these three things before they start to formulate an argument or proof for the general case by asking the following questions:
  - *What do you know is true?* (Only what you are given, or what you can deduce.) *How could you show that on the diagram?*
  - *What do you think might be true but you don’t know for sure?*
  - *What do you want to prove?*

Engaging learners

Classroom techniques for teaching geometric reasoning

Developing geometric reasoning is very challenging for students, requiring risk taking, persistence and the expectation that you will make mistakes and learn from them.

Harnessing students' fascination with scale

People are often fascinated with very large or very small items. We are particularly fascinated with large items that should be small and small items that should be large. For an example of this fascination, follow the link below to the news story about giant marionettes in Perth, WA. An estimated 1.4 million people attended *'The Giants extravaganza'*!

http://www.perthnow.com.au/news/western-australia/giants-in-perth-day-three/story-fnhocxo3-1227220081679

Perth Now
February 2015
Picture: Stewart Allen

To use this story to engage learners we would play one of the news stories, without the audio (at first) and ask students:

What questions do you have?

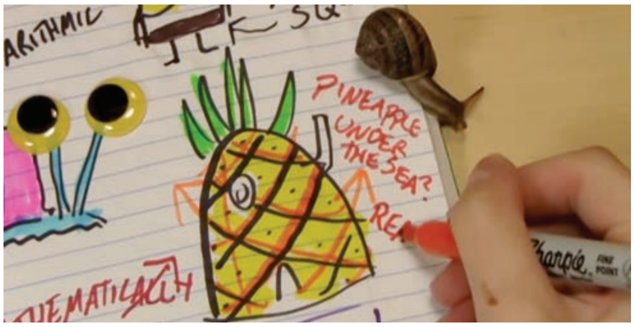
There are films, such as *'The Borrowers'* and *'Gulliver's Travels'*, that play on our fascination with scale. Such films and images can be used to make connections between measurement, scale, enlargement and fractions.

We can support our students to develop a disposition towards using maths in their lives, ie becoming numerate, not only through the use of 'real world' maths problems, but through fostering a disposition towards asking mathematical questions about everything they see. We develop this disposition in our students when we promote, value and share their curiosity and provide opportunities for them to develop their questions and explore solutions to their questions.

Vi Hart's art

Introduce your students to the stimulating work of Vi Hart, an artist, musician and mathematician.

Vi's doodling in mathematics fascinates students with her rapid speech, skilful sketching and unusual explorations into questions such as 'Is SpongeBob SquarePants' home really a pineapple?'



Source: Vi Hart (2012) *Open Letter to Nickelodeon, Re: SpongeBob's Pineapple under the Sea*, accessed at https://www.youtube.com/watch?v=gBxeju8dMho

From tell to ask

Transforming tasks by modelling the construction of knowledge (Examples 1–7)

The idea that education must be about more than transmission of information that is appropriately recalled and applied, is no longer a matter for discussion. We know that in order to engage our students and to support them to develop the skills required for success in their life and work, we can no longer rely on a 'stand and deliver' model of education. It has long been accepted that education through transmission of information has not worked for many of our students. Having said this, our classrooms do not necessarily need to change beyond recognition. One simple, but highly effective strategy for innovation in our classrooms involves asking ourselves the question:

What information do I need to tell my students and what could I challenge and support them to develop an understanding of for themselves?

Students already have an understanding of the everyday usage of the term 'similar' but no amount of reasoning will lead my students to create the mathematical protocols and definitions of similarity, eg the term **scale factor**, for themselves. They need to receive this information in some way. However, it *is* possible my students can be challenged with questions that will result in them identifying the characteristics and properties of similar triangles, so I don't need to instruct that information.

At this stage of development, students can develop an understanding of **what similarity is through visual recognition when generating and comparing similar shapes**. When teachers provide opportunities for students to **identify and describe the properties of similar shapes, they require their students to generalise from the classifications they have made**. Telling students **properties** removes this natural opportunity for students to make conjectures and verify connections that they notice.

Teachers can support students to **identify properties of similarity** by asking questions as described in the **Understanding** proficiency. **What patterns/connections/ relationships can you see?** The intent of this question is to promote learning design that intentionally plans for students to develop a disposition towards looking for patterns, connections and relationships.

When we are feeling 'time poor' it's tempting to believe that it will be quicker to tell our students a definition, rather than ask a question (or series of questions) and support them to establish a definition for themselves. Whether this is true or not really depends on what we have established as our goal. If our goal is to have students recall and apply a particular rule or theorem during the current unit of work, then it probably is quicker to tell them the rule and demonstrate how to apply it. However, when our goal extends to wanting students to develop conceptual understanding, to learn to think mathematically, to have a self-concept as a confident and competent creator and user of mathematics, then telling students the rule is a false economy of time. When we challenge our students to *establish* a theorem, we model that algebra can be powerful and useful. We provide our students with an authentic context for working algebraically. Telling students formulae removes this opportunity for students to generalise.

Curriculum and pedagogy links

The following icons are used in each example:

The '**AC**' icon indicates the Australian Curriculum: Mathematics content description(s) addressed in that example.

The '**Bringing it to Life (BitL)**' tool icon indicates the use of questions from the *Leading Learning: Making the Australian Curriculum Work for Us* resource.

The Bringing it to Life tool is a questioning tool that supports teachers to enact the AC: Mathematics Proficiencies: http://www.acleadersresource.sa.edu.au/index.php?page=bringing_it_to_life

The '**From tell to ask**' icon indicates a statement that explains the transformation that is intended by using the task in that example.

This idea of moving 'From tell to ask' is further elaborated (for Mathematics and other Australian Curriculum learning areas) in the 'Transforming Tasks' module on the *Leading Learning: Making the Australian Curriculum work for Us* resource: http://www.acleadersresource.sa.edu.au/index.php?page=into_the_classroom

Look out for the purple pedagogy boxes, that link back to the SA TfEL Framework.

From tell to ask examples	
Example 1: It looks the same but is it similar? Students use the enlargement transformation to explain similarity and develop the conditions for triangles to be similar	ACMMG220 ◆ ◆
Example 2: Clones and mutants Students use the enlargement transformation to explain similarity and develop the conditions for triangles to be similar	ACMMG220 ◆ ◆
Example 3: Paper sizes – Ratios of similar rectangles Students solve problems using ratio and scale factors in similar figures	ACMMG221 ◆
Example 4: Enlargement and squared and cubed attributes Students solve problems using ratio and scale factors in similar figures	ACMMG221 ◆
Example 5: Kite in a square – Many ways to reason Students solve problems using ratio and scale factors in similar figures	ACMMG221 ◆
Example 6: Proving similarity Students use the enlargement transformation to explain similarity and develop the conditions for triangles to be similar	ACMMG220 ◆ ◆
Example 7: Is it congruent? Students solve problems using ratio and scale factors in similar figures	ACMMG221 ◆

Example 1: It looks the same but is it similar?

ACMMG220 ◆ ◆

Students use the enlargement transformation to explain similarity and develop the conditions for triangles to be similar

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/ relationships can you see?

Can you represent/calculate in different ways?

Reasoning proficiency:

In what ways can you communicate?

In what ways can your thinking be generalised?

Instead of ***telling*** students about similarity, we can challenge students to recognise the relationships between similar and congruent shapes for themselves, by ***asking*** questions.

Students will often have a very strong sense of whether shapes are similar because it is a word used in everyday language. While this prior knowledge will support the conceptual understanding of the mathematical property of similarity, we must provide students with the opportunity to consider how the two uses are the same and how are they different.

We can start by asking students by thinking of situations where they might use the word *similar* or use images such as these (or ones students have taken themselves of things that they consider similar). Ask students:

- *What are the different ways we use the word 'similar' to describe things?*

Sometimes things are different colours or have a few features that are the same. When we use the word similar in mathematics, we need to be more precise so that everyone has the same understanding.

Provide students with a range of different shapes which are mathematically similar, then ask:

- *What's the same about the shapes/objects?*
- *What's different about the shapes/objects?*
- *What connections can you see?*
- *Can you write a definition for a Year 8 student so that they could judge whether shapes are similar or not?*
- *How is your definition the same or different to the one in the maths dictionary?*
- *What is the difference between 'similar' and 'congruent'? Can shapes be 'similar' and 'congruent'?* (For geometrical shapes to be similar the angles must be the same and the sides must be in the same ratio. Congruent shapes are similar shapes with the ratio of the sides being 1:1.)
- *Can you order shapes that are similar in some way? How might you describe the relative size of these shapes?*

Encourage students to make a multiplicative comparison between the shapes, such as, the sides are 1½ times as long, rather than additive, such as, this side is 4 cm longer, you just add on half the length. Ask students:

- *What might 'twice as big' mean?* (When the scale factor is 2, the side lengths and hence perimeter is doubled, but the area of planar figures is squared (x 2^2) and the volume of a 3D solid is (x 2^3). See 'Example 4: Enlargement and squared and cubed attributes'.)

Example 2: Clones and mutants

ACMMG220 ◆ ◆

Students use the enlargement transformation to explain similarity and develop the conditions for triangles to be similar

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/ relationships can you see?

Can you represent/calculate in different ways?

Reasoning proficiency:

In what ways can you communicate?

In what ways can your thinking be generalised?

Instead of ***telling*** students about similarity, we can challenge students to recognise the relationships between similar and congruent shapes for themselves, by ***asking*** questions.

Ask students to design their own simple plane shape and cut it out of card. You may wish to place restrictions on the shape such as it must fit inside a 20 cm square, have straight sides, or a given number of sides.

Ask them to make at least 2 different 'clones' of their shape by rotating, reflecting, translating or a combination of these transformations, tracing the image and cutting it out.

- *Why do you think they might be referred to as 'clones'? In what sense might these shapes be considered to be 'genetically identical individuals'?*
- *What do you notice when you rotate, reflect, translate it? What do you think changes/remains the same about the shape?*

Students at this stage are familiar with the transformations of rotation (turn), reflection (flip) or translation (slide). Teachers can challenge their students to adapt the known to the unknown through questioning.

The gene that determines size of shapes in our classroom mutates due to radiation. Create a 'mutant' of your shape.

- *How will you change the size without changing other features of the shape? What other features do you need to keep the same?*

Students will quite often be very creative in the methods they use. If their shape is complex and they are finding it challenging, support them through questioning to halve or double the lengths of the sides but keep the orientation the same by making sure the mutant sides are parallel to the original shape.

Place all the students' shapes in a box and get each one in turn, to draw out 5 and sort them into groups on the board, explaining and demonstrating why they think they belong together (congruent) or whether they are 'mutants' (in this case similar).

'Clones' is not a mathematical term. Mathematicians refer to these as **congruent** shapes.

- *What does congruent mean? How could we find out?*

'Mutants' is not a mathematical term. Mathematicians refer to these as **similar** shapes.

- *What does similar mean in everyday language? What might similar mean in mathematical language? How could we find out?*
- *What is the same? What is different about the similar shapes?*

The exploration into the properties of similar shapes and scale factors used in 'Example 1: It looks the same but is it similar?' would be appropriate here.

Example 3: Paper sizes – Ratios of similar rectangles

ACMMG221 ◆

Students solve problems using ratio and scale factors in similar figures

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/ relationships can you see?

Can you represent/calculate in different ways?

Reasoning proficiency:

In what ways can your thinking be generalised?

What can you infer?

Instead of ***telling*** students about similarity, we can challenge students to recognise the relationships between similar shapes for themselves, by ***asking*** questions.

Begin a discussion about paper sizes the students may be familiar with. Ask students:

- *What different paper sizes are you familiar with? What are they used for? Why?*

Issue the students with pieces of an A3, A4 and A5 sheet of paper, scissors, rulers and sticky tape. Ask students:

- *What is the same/different about these pieces of paper?* (Students often identify an additive relation between the different sizes, for example, 'The difference goes up each time'. Encourage students to find a multiplicative relationship.)
- *If you know the measurements of these three sheets, what might be the size of A2 and A6 sheets? Why?*
- *What might be an estimate of the size (dimensions) and area of an A0 size sheet? How could you check?*
- *What is the scale factor for the enlargement for A5 to A4, A4 to A3? Can you generalise the relationship between 'consecutive' paper sizes, for example, An and A(n+1)?* (The area doubles but the scale factor is $\sqrt{2}$. This concept is encountered in both Example 1 and Example 4 and is a good connection with irrational numbers.

For students who are seeking an intellectual challenge, ask:

- *Are paper sizes the same all over the world?*
- *What is the difference/same about paper sizes A, B and C? Are they similar rectangles? Can you generalise the ratios between the paper sizes B and C?*

Links to authentic contexts

Using examples/analogies from real life situations engages students and can contextualise the purpose of a mathematical process. Students are most likely to be familiar with the A5, A4 and A3 sizing of paper but not been aware of the fact that they are similar rectangles.

Example 4: Enlargement and squared and cubed attributes

ACMMG221 ◆
Students solve problems using ratio and scale factors in similar figures

Questions from the BitL tool

Understanding proficiency:
What patterns/connections/ relationships can you see?

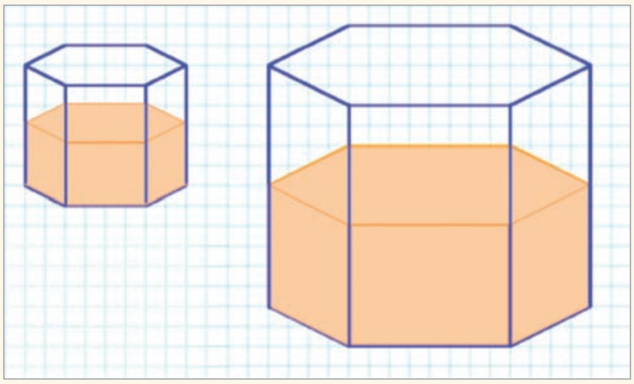
Reasoning proficiency:
In what ways can your thinking be generalised?

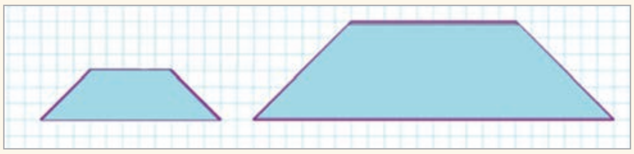
What can you infer?

Instead of ***telling*** students about similarity, about the ratios that exist between similar shapes, we can challenge students to recognise the relationships for themselves, by ***asking*** questions.

For each of these pairs of images below, ask students:

- *What's the same about the shapes/objects?*
- *What's different about the shapes/objects?*
- *What connections can you see between their measurements?*





In this example, each of the initial shapes has been enlarged with a scale factor of 2. A scale factor of 2 will mean that the surface area is 4 times the original (that's 2^2) and the volume is 8 times the original (that's 2^3). Noticing connections between one shape or object and an enlarged version of it, opens a conversation where teachers can move students on from an understanding of congruence (identical in shape and size) to similarity (where the only difference between the shapes is scaling). This context also connects with the nature of the formulae for perimeter, area and volume in measurement.

Teachers can challenge students to generalise their findings, by asking:

- *What if you used a different scale factor… will the relationships be the same?*
- *Can you write a rule (algebraically) to express the relationship between scale factor and perimeter?*
- *Can you write a rule (algebraically) to express the relationship between scale factor and area?*
- *Can you write a rule (algebraically) to express the relationship between scale factor and volume?*

Example 5: Kite in a square – Many ways to reason

ACMMG221 ◆

Students solve problems using ratio and scale factors in similar figures

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/ relationships can you see?

Can you represent/calculate in different ways?

Reasoning proficiency:

In what ways can your thinking be generalised?

In what ways can you communicate?

Instead of ***telling*** students about the ratios that exist between similar shapes, we can challenge students to recognise the relationships for themselves, by ***asking*** questions.

In this activity from the NRICH website, students are provided three different ways of proving the same result. The complete solutions are jumbled up, so that students can engage with the geometrical proofs without having to start from scratch.

Provide students with enough time to tackle this problem before introducing the resources provided with this activity.

The link to this problem on the NRICH site is: http://nrich.maths.org/8301/note

The NRICH website contains a large collection of high quality maths problem solving tasks, together with suggestions about content that may be related to the task, ways to get started and different (valid) solutions that have been submitted by students from around the world.

Connecting the learning: Making links

Activities like this one from the NRICH website, show that we can design rich experiences that:

- show there is more than one way to solve a problem and in this case, using different branches of mathematics, for example, coordinate geometry, similar triangles and Pythagoras' Theorem
- show that the different branches of mathematics lead to the same result making strong connections between their learning, reassuring them of the consistency of mathematics
- provide consolidation and practice of previous learning that verifies the results obtained by the new knowledge
- model formal written protocols used to communicate geometric reasoning by creating a jigsaw for students to reconstruct.

Example 6: Proving similarity

ACMMG220 ◆ ◆

Students use the enlargement transformation to explain similarity and develop the conditions for triangles to be similar

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/ relationships can you see?

Can you represent/calculate in different ways?

Reasoning proficiency:

In what ways can you communicate?

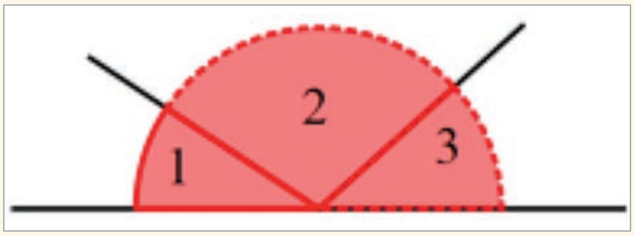
What can you infer?

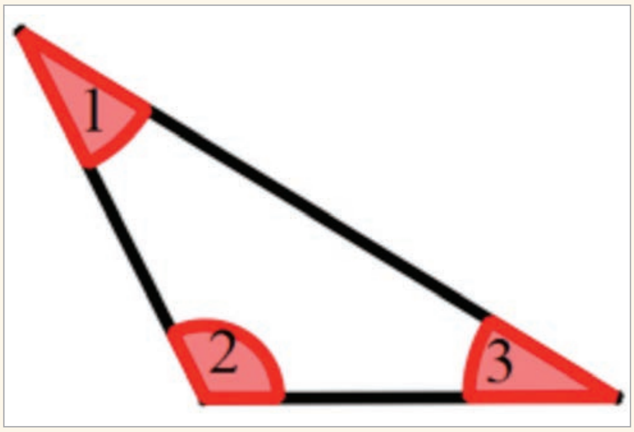
Instead of ***telling*** students about similarity, we can challenge students to recognise the relationships between similar and congruent shapes for themselves, by ***asking*** questions.

When proving triangles were congruent in Year 8, we discovered that we did not have to prove that all sides and all angles were equal. This is an opportunity to use the manipulatives students used to develop the 4 conditions for proof, as a demonstration for full class discussion, for example, SSS, SAS, AAcorrS and RHS (see DECD, 2017, *Geometric reasoning: Year 8 – Mathematics Conceptual Narrative*).

Elaborate by asking students:

- *Are there a minimum number of conditions to prove triangles are similar? What do you think?* (From their congruence proof, students may remember that they do not need to know all three angles are the same. If two are the same then the third must be, as all angles in a triangle must add up to 180°. This is an opportunity for students to demonstrate their geometric reasoning.)
- *Is it possible to have all three angles the same and not have the sides in the same ratio? How could we check this?* (Cut any three angles at a straight edge and use these to construct a triangle.)





- *Can you construct a triangle that is not similar to the others? How will you check?* (Students might realise that they don't need the third when constructing triangles which reinforces the concept that only 2 angles need to be the same. A visual check will suggest the similarity but we should encourage the students to provide evidence of the other characteristics of similar triangles by measuring the sides and checking that the ratio is the same. We should also encourage students to generalise this result.)
- *Will this always be the case if we keep the angles the same? Does it make sense?*
- *What happens when we double the length of one side? What do you notice?* (While a generalised discussion is not a formal proof, it is encouraging students to think more deeply about why the ratio will remain the same.)

Example 7: Is it congruent?

ACMMG221 ◆

Students solve problems using ratio and scale factors in similar figures

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/relationships can you see?

Can you answer backwards inverse questions?

Reasoning proficiency:

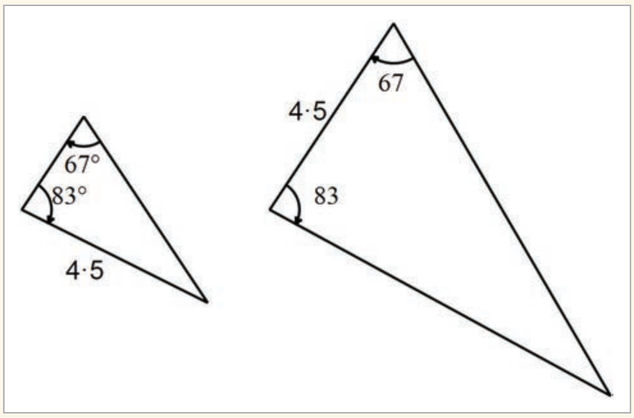
In what ways can you communicate?

What can you infer?

Instead of ***telling*** students about the ratios that exist between similar shapes, we can challenge students to recognise the relationships for themselves, by ***asking*** questions.

Year 8 students have explored the conditions for triangles being congruent (SSS, SAS, AAcorrS and RHS). When students have identified that both triangles have 2 angles and a side that are the same (ie AA **corresponding** S) it is common that they do not confirm that the side is in a corresponding position on both triangles. If it is not a corresponding or matching side, the triangles will be similar but not congruent.

Present one student's thinking about the diagram below to the class:



'I don't get it. These triangles must be congruent because they have two angles the same and one of the sides is the same. We learned that in Year 8 but these look alike but they aren't identical. Why aren't they congruent?'

- *What's wrong with this thinking?*

Challenging student misconceptions: Presenting other people's examples

Asking questions of the form: *What's wrong with…?* or *Why is it not…?* or *Why can't I…?* are a good way of hearing a student's reasoning. These questions also provide us with an opportunity to challenge students' thinking if we compile a number of examples that reflect students' possible misconceptions. The examples don't have to be presented as their mistakes, just as examples of thinking that is commonly seen when people are learning to work with *similar* and *congruent* triangles. Once the students have identified the issues they can reflect on the effect that it has had on their own thinking.

Proficiency: Problem Solving

Proficiency emphasis and what questions to ask to activate it in your students (Examples 8–12)

Problems are described in the AC: Mathematics as 'meaningful' or 'unfamiliar'. Students of *all* abilities and ages should be provided with experiences of both meaningful and unfamiliar problems.

Meaningful problems

Meaningful problems are set in a context that a student can project themselves into. It may be that the mathematics and strategy being applied is familiar to the student or the problem relates to their own life experience. Connecting with a context does not mean that the students have to see it as 'fun' nor does it have to relate to an immediately practical situation from daily life.

Students can connect with a problem through provocations such as the use of manipulatives (either physical or digital) or through a dramatisation (eg, a story, interesting background information, a video clip). The intention is to give students the opportunity to work as a mathematician would work, in a context that they can access at their current stage of development.

Unfamiliar problems

Fundamentally there are two groups of unfamiliar problems:

- Problems for which the students would not be able to say that they had done a similar example previously, they would therefore need to create an approach (develop a strategy).
- Problems in which the students develop a new piece of knowledge. They begin the problem by applying the knowledge/skills that they have and they complete the problem having recombined that knowledge to form a new piece of understanding.

Growth mindset: Learning that not knowing is the beginning of a learning opportunity

Unfamiliar problems tend to provoke a response of, 'I don't know', or 'I'm not sure'. Students respond differently to this feeling, some shut down, others begin to ask, 'But how could I work that out?'

In developing powerful learners we are aiming for all of our students to learn that 'not knowing' is the *beginning of a learning opportunity* and that the first move that they need to make on the journey to finding out more is to ask, 'What could I do to work this out?'

Engaging in problem solving supports the move *from tell to ask*

Instead of ***telling*** students:

- the problem to solve
- the information they'll need
- the steps they should take.

We can ***ask students to identify***:

- the problem to solve
- the information they'll need
- a possible process to use.

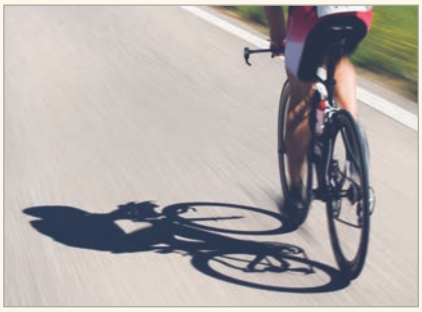
Proficiency: Problem Solving examples	
Example 8: Shadows Students solve problems using ratio and scale factors in similar figures	ACMMG221 ◆
Example 9: Federation Square Students solve problems using ratio and scale factors in similar figures	ACMMG221 ◆
Example 10: Stretching to enlarge Students solve problems using ratio and scale factors in similar figures	ACMMG221 ◆
Example 11: Morphing an alien on a coordinate grid Students solve problems using ratio and scale factors in similar figures	ACMMG221 ◆
Example 12: Growing squares Students solve problems using ratio and scale factors in similar figures	ACMMG221 ◆

Example 8: Shadows

ACMMG221 ◆
Students solve problems using ratio and scale factors in similar figures

Questions from the BitL tool
Problem solving proficiency:
Interpret; Model and plan; Solve and check; Reflect.
Reasoning proficiency:
What can you infer?

Instead of ***telling*** students the information they'll need and the steps they should take, we can ***ask*** them to identify the information they'll need and the steps they could take.



Provide students with a range of images with shadows (such as those above), or ask students to take some of their own.

Ask students:

- *What's the first question that comes to mind?* (Knowing the height of one of two objects and the length of the shadows might be used to determine the height of an unknown object, such as, a tree in the yard or a school building.)

Interpret

What question have you selected? What might you need to work out to answer that question? What information is helpful? What information is not useful? What extra information do you need? What information can you reasonably infer? (Remember that telling students or prompting heavily is STEALING the opportunity for them to realise for themselves later on in the task.)

Model and plan

Do you have an idea? How might you start? Would it help if you thought about a similar problem for a simpler figure first? (Ask students to speak to someone who they think is being a good problem solver today and ask them to show them what they are trying.)

Solve and check

What height for the person is too high? What height is too low?

Questions to be used only after students have grappled with the problem for a few minutes:

Does that seem right to you? Do other people think that too? Could you give a range of possible values? Can you estimate where the light source is, where the wall is? What could you do to check your thinking?

Reflect

Ask students to pair up with someone who did it differently to discuss:

How do your methods compare? What do you like about each other's strategy? How could you help each other to improve? Have you reached the same or a similar conclusion? How efficient was your strategy? Is there something that you would do differently next time? Is there a more efficient way?

Example 9: Federation Square

ACMMG221 ◆
Students solve problems using ratio and scale factors in similar figures

Questions from the BitL tool
Problem solving proficiency:
Interpret; Model and plan; Solve and check; Reflect.
Reasoning proficiency:
What can you infer?

Instead of ***telling*** students the information they'll need and the steps they should take, we can ***ask*** them to identify the information they'll need and the steps they could take.

This activity could be considered a ***Three-Act Maths Task*** like those on Dan Meyer's blog: 101 questions. It can be presented to students along with the question, *What's the first question that comes to mind?*

Identifying the question to solve

The group can share questions and sort them into mathematical and non-mathematical questions. Then, of the mathematical questions, students can sort their questions into those that cannot be answered with the given information and those that could be answered using the given information or additional information that could be inferred.

Dan Meyer has a technique, that we have seen many teachers adopt when generating and collecting questions from students. First he asks students to individually write down questions that come to mind. Then, as he invites students to share their questions, he writes students' names next to the questions. He also asks if anyone else likes that question. *'Did you write it down, or if you didn't perhaps you still think that it's a good question.'* Through doing this, both Dan and his class get a sense of the questions that are of interest to the students.

Begin by asking if any students have visited Federation Square in Melbourne and discuss the purpose and history of the space with them. Ask whether anyone had noticed the unique sandstone building façades. Show a picture of the façade that has not been marked.

Source: http://fedsquare.com/about/history-design

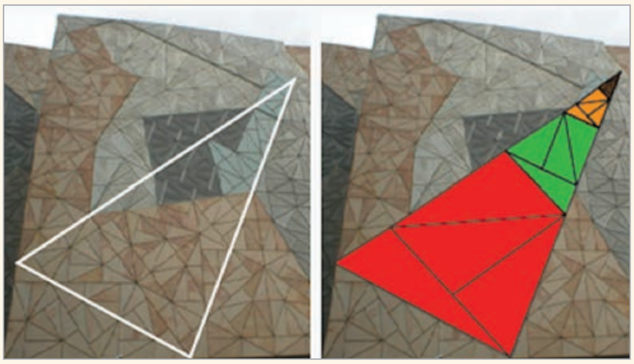
- *What's the first question that comes to mind?* (Is the pattern random? What shapes is the wall made up of? Are the triangles similar or congruent? How many triangles are on the wall? …on a section of the wall?)

Keeping control of the question

If students' questions sit outside of the area that we want them to work in, we have some choices to make. We can always add our own question to the list and ask for that question to be solved. But we probably want to minimise that, as students may lose interest in generating possible questions if they know that we'll always replace their questions with our own.

Students may have generated variations on the question that we had intended and if we can see that they will still use the concepts that we had intended, we could either let them answer their own question or ask them to answer our question and then reflect on whether or not their question was also answered in the process.

Show students the following images:



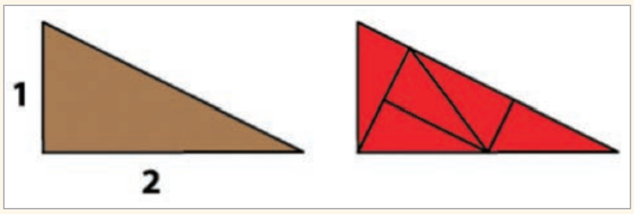
Used with permission and adapted from Polster B & Ross M (2012) 'Federation Square's hidden gem', *The Age*, 16 July edition, retrieved from http://www.qedcat.com/tempMMindex.html

Then ask the following questions:

- *How is this pattern an example of congruent triangles inside a similar triangle?*

- *What might you need to work out to answer that question? What information is helpful? What information is not useful? What extra information do you need? What information can you reasonably infer?* (Remember that telling students or prompting heavily is STEALING the opportunity for them to realise for themselves later on in the task.)

To challenge students' thinking, you could ask them to identify congruent triangles and then find a larger triangle that contains 5 congruent triangles.



This triangular piece can be divided into 5 congruent triangles, similar to the original triangle.

- Construct a right angled triangle with the given proportions and divide it into 5 congruent triangles. (This will need to be large to allow for two more such divisions.)
- Divide each of these five triangles into five and then each of those into five to recreate the tiling.

This process can be applied to other figures.

An extended task relating to the tiles in Federation Square can be found in the Mathematical Association of South Australia (MASA) publication, ***Stage 1 Folio Tasks – SACE*** (available at http://www.masanet.com.au/publications-booklet/).

Example 10: Stretching to enlarge

ACMMG221 ◆

Students solve problems using ratio and scale factors in similar figures

Questions from the BitL tool

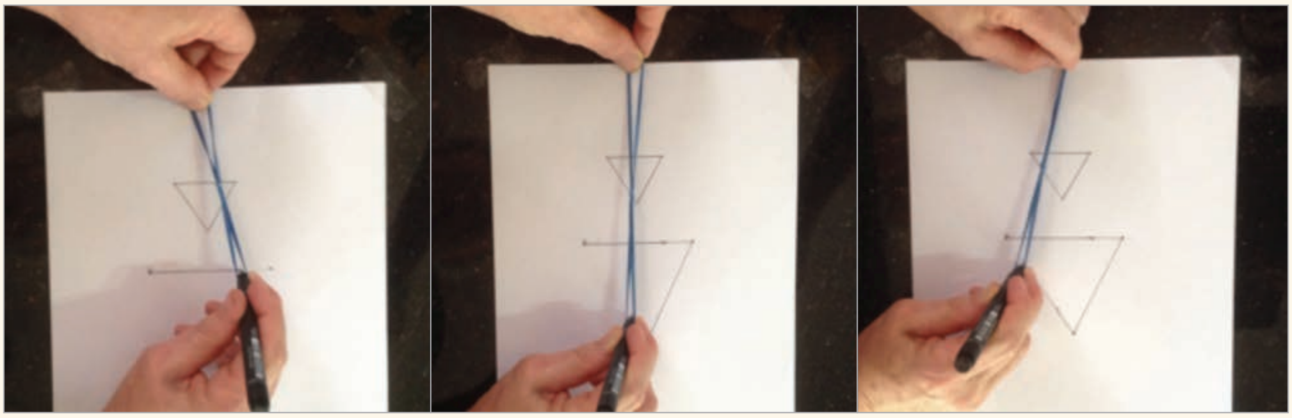
Problem solving proficiency:

Interpret; Model and plan; Solve and check; Reflect.

Reasoning proficiency:

What can you infer?

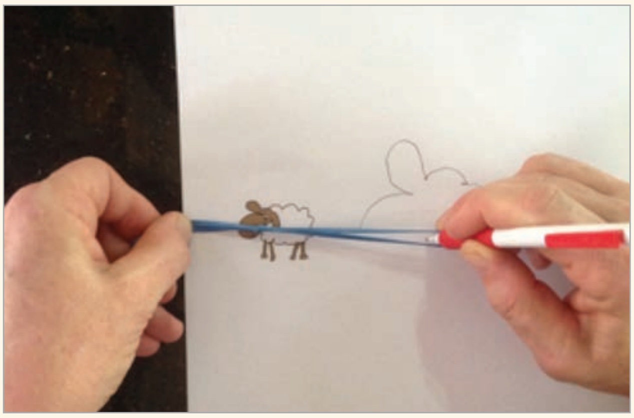
Instead of ***telling*** students the information they'll need and the steps they should take, we can ***ask*** them to identify the information they'll need and the steps they could take.



In this activity students can enlarge an original shape using a large piece of paper, a pen and several elastic bands. (Consider all safety issues in relation to the use of this activity.)

Interlink two elastic bands. Anchor one of the linked bands at point to one side of the image and place the pen inside the second linked band.

Stretch the bands until the knot in the middle aligns with a point on the original shape. Place the pen on the paper and move it so the knot traces around the shape.



Ask students:

- *What do you notice about the image?* (They appear similar.)
- *How could you check that? How could you work out the scale factor? Would it always work?*
- *What do you think might happen if you linked 3, 4 or more bands together?*

Example 11: Morphing an alien on a coordinate grid

ACMMG221 ◆

Students solve problems using ratio and scale factors in similar figures

Questions from the BitL tool

Problem solving proficiency:

Interpret; Model and plan; Solve and check; Reflect.

Reasoning proficiency:

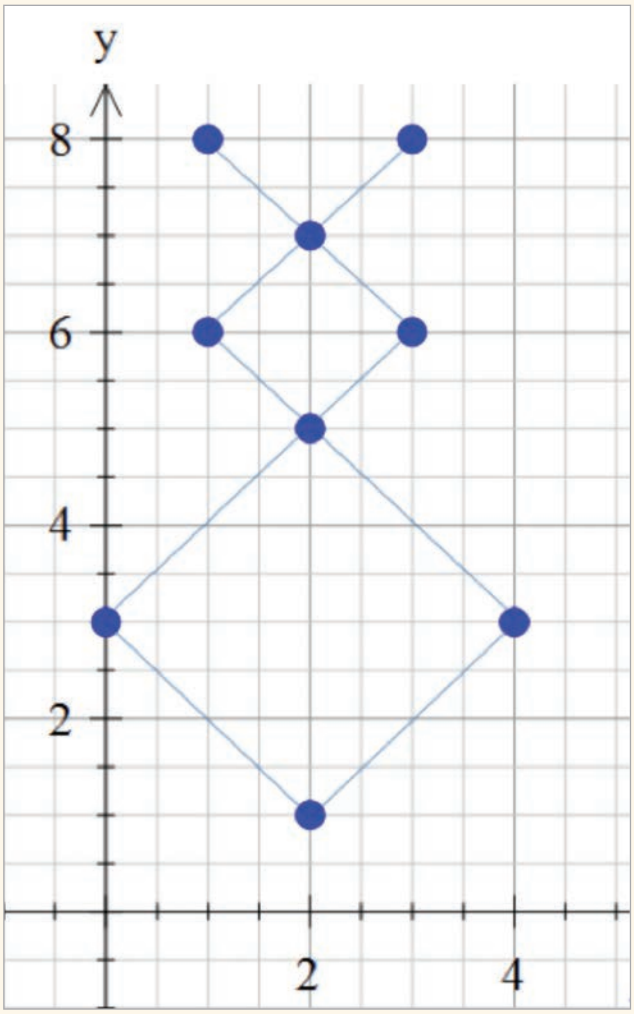
What can you infer?

Instead of ***telling*** students the information they'll need and the steps they should take, we can ***ask*** them to identify the information they'll need and the steps they could take.

In this activity, students plot **SAM**—the **Space Alien Morpher**—using the example coordinates given below, or plot one on their own creation.

Example coordinates:

(2,1) (4,3) (0,3) (2,5) (1,6) (3,6) (2,7) (3,8) (1,8)



To see SAM undergo Morph 1, double both the X and Y coordinates and plot SAM after Morph 1.

Ask students:

- *What's the first question that comes to mind?*
- *What do you notice about the SAM 1?*
- *How could you check that? How could you work out the scale factor? Would it always work?*
- *What do you think might happen if you designed a different Morph? Will SAM always be similar to original self?* (Students can, for example, halve coordinates, multiple by negative numbers, or use a different rule for the X and Y coordinates.)

This also an opportunity for students to apply their learning about gradients, midpoints and distances to gain more information about SAM and the Morphs.

Ask students to pair up with someone who did it differently to discuss:

- *How do your methods compare? What do you like about each other's strategy? How could you help each other to improve? Have you reached the same or a similar conclusion?*

Example 12: Growing squares

ACMMG221 ◆

Students solve problems using ratio and scale factors in similar figures

Questions from the BitL tool

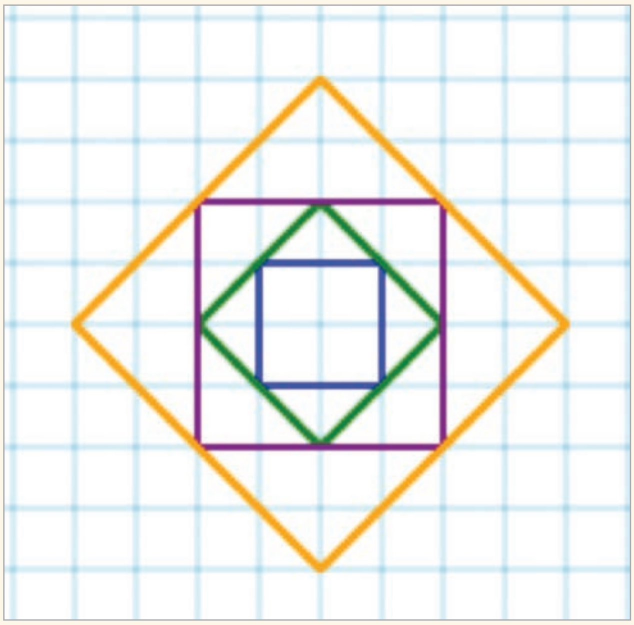
Problem solving proficiency:

Interpret; Model and plan; Solve and check; Reflect.

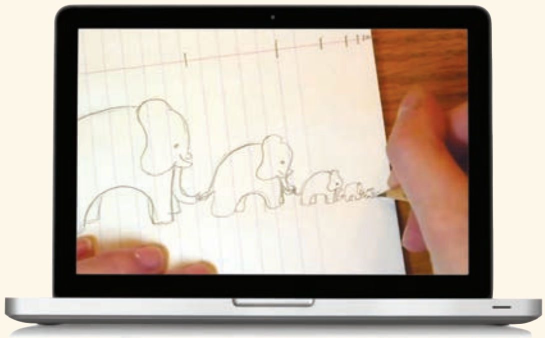
Reasoning proficiency:

What can you infer?

Instead of ***telling*** students the information they'll need and the steps they should take, we can ***ask*** them to identify the information they'll need and the steps they could take.



This type of visual patterning, leading to an infinite series can be extended by considering the thinking of Vi Hart. She is an engaging artist, musician and mathematician who 'doodles' mathematically. One of her videos shows an infinite series of elephants.



Source: Vi Hart (2010) *Doodling in Math Class: Infinity Elephants*, https://www.youtube.com/watch?v=DK5Z709J2eo

Show students the image above and ask:

- *What connections do you see between the squares in this pattern?*
- *Can you demonstrate your thinking numerically?*
- *Can you prove your thinking algebraically?*
- *What if this was 3 dimensional? What connections would there be between the surface area of the boxes? What connections would there be between the volume of the boxes?*

Connections between ‘geometric reasoning’ and other maths content

There are many opportunities to connect to other content in the AC: Mathematics, when we use geometric reasoning as a starting point.

Here are just some of the possible connections that can be made:

Mathematics: Year 9	
Whilst working with geometric reasoning, connections can be made to:	**How the connection might be made:**
Students define congruence of plane shapes using transformations. ACMMG200	Refer to: **Example 2: Clones and mutants** **Example 7: Is it congruent?**
Students develop the conditions for congruence of triangles. ACMMG201	Refer to: **Example 7: Is it congruent?**
Students solve problems involving direct proportion. ACMNA208	Refer to: **Example 1: It looks the same but is it similar?** **Example 2: Clones and mutants** **Example 3: Paper sizes – Ratios of similar rectangles** **Example 4: Enlargement and squared and cubed attributes** **Example 5: Kite in a square – Many ways to reason** **Example 6: Proving similarity** **Example 7: Is it congruent?** **Example 8: Shadows** **Example 9: Federation Square** **Example 10: Stretching to enlarge.**
Students find the distance between two points located on a Cartesian plane using a range of strategies, including graphing software. ACMNA214 Students find the midpoint and gradient of a line segment (interval) on the Cartesian plane using a range of strategies, including graphing software. ACMNA294	Refer to: **Example 11: Morphing an alien on a coordinate grid.**
Students solve problems involving surface area and volume of right prisms. ACMMG218	Refer to: **Example 12: Growing squares.**

Making connections

We know that when students meet a concept frequently, and in different contexts, they have a greater chance of developing understanding. With this in mind, it is our responsibility to help our students to make these connections by intentionally designing tasks that connect a number of different content descriptions. Alternatively, connections can be made through questioning individuals or small groups of students.

‘Geometric reasoning’ from Foundation to Year 10A

The AC: Mathematics year level content descriptions shown here have been colour coded to highlight the following curriculum aspects of working with geometric reasoning.

Estimate/measure/compare/identify/classify using geometrical reasoning ◆
In Foundation to Year 2 through the ‘shape’ content descriptions, students sort/recognise/name/describe/draw/classify. In Year 3 to Year 5 students estimate/measure/compare/identify/classify using geometrical reasoning.

Investigate and generalise using geometrical reasoning ◆ ◆
In Year 6 to Year 8 students mostly investigate and solve.

Solve and prove using geometrical reasoning ◆
In Year 9 and Year 10 students mostly solve and prove by applying logical reasoning.

Year level	‘Shape’ content descriptions from the AC: Mathematics
Foundation ◆	Students sort, describe and name familiar two-dimensional shapes and three-dimensional objects in the environment. ACMMG009
Year 1 ◆	Students recognise and classify familiar two-dimensional shapes and three-dimensional objects using obvious features. ACMMG022
Year 2 ◆	Students describe and draw two-dimensional shapes, with and without digital technologies. ACMMG042
Year 2 ◆	Describe the features of three-dimensional objects. ACMMG043
Year level	**‘Geometric reasoning’ content descriptions from the AC: Mathematics**
Year 3 ◆	Students identify angles as measures of turn and compare angle sizes in everyday situations. ACMMG064
Year 4 ◆	Students compare angles and classify them as equal to, greater than or less than a right angle. ACMMG089
Year 5 ◆	Students estimate, measure and compare angles using degrees. Construct angles using a protractor. ACMMG112
Year 6 ◆ ◆	Students investigate, with and without digital technologies, angles on a straight line, angles at a point and vertically opposite angles. Students use results to find unknown angles. ACMMG141
Year 7 ◆	Students identify corresponding, alternate and co-interior angles when two straight lines are crossed by a transversal. ACMMG163
Year 7 ◆ ◆	Students investigate conditions for two lines to be parallel and solve simple numerical problems using reasoning. ACMMG164
Year 7 ◆ ◆	Students demonstrate that the angle sum of a triangle is 180° and use this to find the angle sum of a quadrilateral. ACMMG166
Year 7 ◆	Students classify triangles according to their side and angle properties and describe quadrilaterals. ACMMG165
Year 8 ◆	Students define congruence of plane shapes using transformations. ACMMG200
Year 8 ◆	Students develop the conditions for congruence of triangles. ACMMG201
Year 8 ◆ ◆	Students establish properties of quadrilaterals using congruent triangles and angle properties, and solve related numerical problems using reasoning. ACMMG202
Year 9 ◆ ◆	Students use the enlargement transformation to explain similarity and develop the conditions for triangles to be similar. ACMMG220

Year level	'Geometric reasoning' content descriptions from the AC: Mathematics *continued*
Year 9 ◆	Students solve problems using ratio and scale factors in similar figures. ACMMG221
Year 10 ◆	Students formulate proofs involving congruent triangles and angle properties. ACMMG243
Year 10 ◆	Students apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes. ACMMG244
Year 10A ◆	Students prove and apply angle and chord properties of circles. ACMMG272

Numeracy continuum: Using spatial reasoning	
End Foundation	Visualise 2D shapes and 3D objects: sort and name simple 2D shapes and 3D objects.
End Year 2	Visualise 2D shapes and 3D objects: identify, sort and describe common 2D shapes and 3D objects.
End Year 4	Visualise 2D shapes and 3D objects: visualise, sort, identify and describe symmetry, shapes and angles in the environment.
End Year 6	Visualise 2D shapes and 3D objects: visualise, sort, describe and compare the features of objects such as prisms and pyramids in the environment.
End Year 8	Visualise 2D shapes and 3D objects: visualise, describe and apply their understanding of the features and properties of 2D shapes and 3D objects.
End Year 10	Visualise 2D shapes and 3D objects: visualise, describe and analyse the way shapes and objects are combined and positioned in the environment for different purposes.

Source: ACARA, Australian Curriculum: Mathematics, Version 8.1

Resources

NRICH website

http://nrich.maths.org

In this conceptual narrative we have highlighted the possibility of using tasks from an organisation called NRICH enriching mathematics.

The NRICH website contains a large collection of high quality maths problem solving tasks, together with suggestions about content that may be related to the task, ways to get started and different (valid) solutions that have been submitted by students from around the world.

Each time we refer to one of the NRICH resources we have provided you with a link to that activity.

Dan Meyer's blog: 101 questions

http://www.101qs.com

Dan's blog contains images and short films that can be presented to students along with the question: *What's the first question that comes to mind?*

A spreadsheet of ***Dan Meyer's Three-Act Maths Tasks*** can be accessed at http://bit.ly/DM3ActMathTasks.

Notes

Do you want to feel more confident about the maths you are teaching?

Do you want activities that support you to embed the proficiencies?

Do you want your students thinking mathematically rather than just doing maths?

If you answered **yes** to any of these questions, then this resource is for you.

Packed full of examples, along with questions you can ask students as they engage in their learning, this resource supports you to develop confidence in teaching the Australian Curriculum: Mathematics.

Produced by Office for Education and Early Childhood

Excluded from NEALS