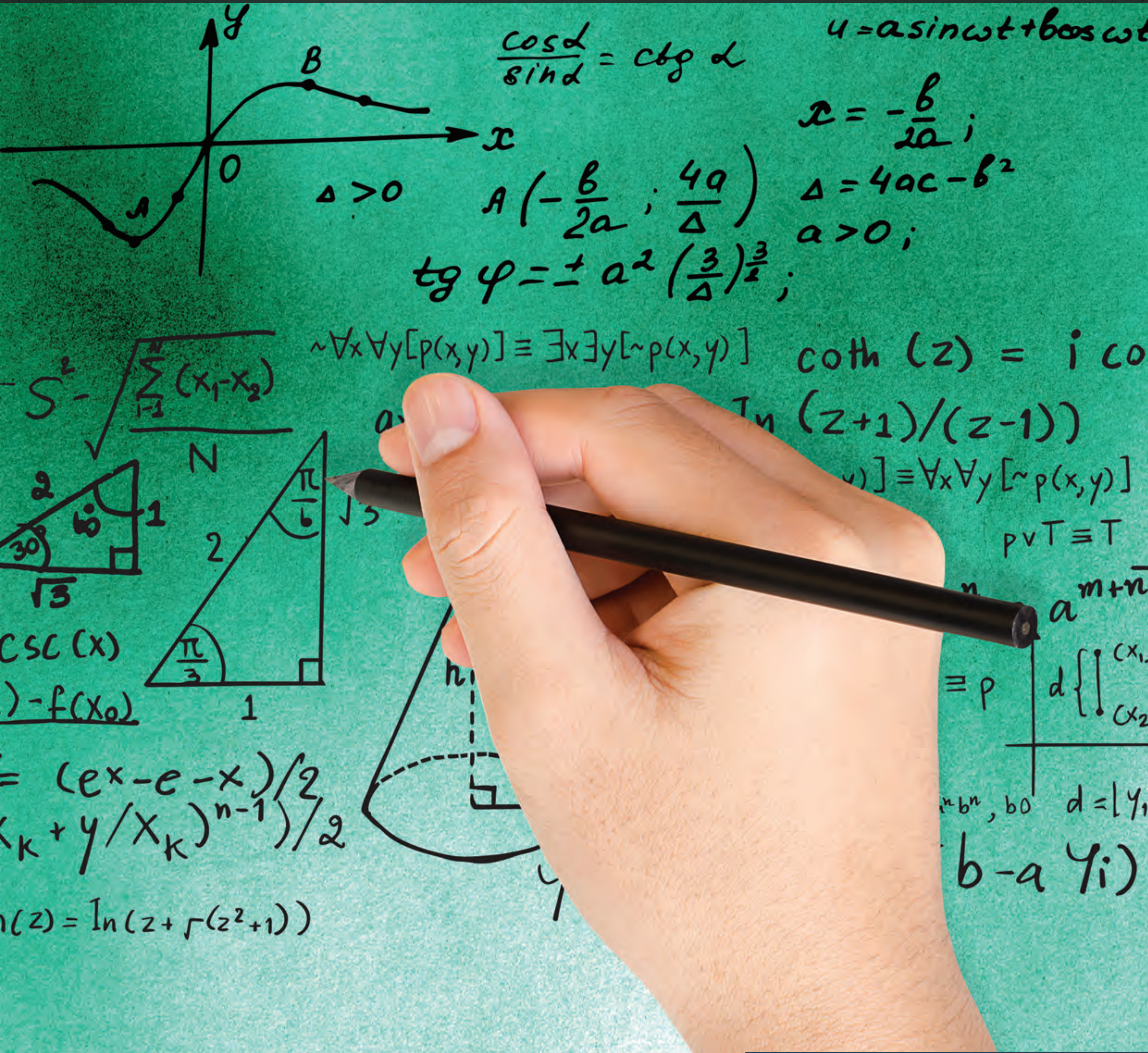
Pythagoras and trigonometry: Year 9

MATHEMATICS CONCEPTUAL NARRATIVE

Leading Learning: Making the Australian Curriculum work for us
by bringing CONTENT and PROFICIENCIES together



www.acleadersresource.sa.edu.au

Government of South Australia
Department for Education and Child Development

Contents

Resource key

The '**AC**' icon indicates the Australian Curriculum: Mathematics content description(s) addressed in that example.

The '**From tell to ask**' icon indicates a statement that explains the transformation that is intended by using the task in that example.

More information about 'Transforming Tasks':
http://www.acleadersresource.sa.edu.au/index.php?page=into_the_classroom

Look out for the purple pedagogy boxes, that link back to the SA TfEL Framework.

The '**Bringing it to Life (BitL)**' tool icon indicates the use of questions from the *Leading Learning: Making the Australian Curriculum Work for Us* resource.

Bringing it to Life (BitL) key questions are in bold orange text.

Sub-questions from the BitL tool are in green medium italics – these questions are for teachers to use directly with students.

More information about the 'Bringing it to Life' tool:
http://www.acleadersresource.sa.edu.au/index.php?page=bringing_it_to_life

Throughout this narrative—and summarised in **'Pythagoras and trigonometry' from Year 9 to Year 10A** (see page 33)—we have colour coded the AC: Mathematics year level content descriptions to highlight the following curriculum aspects of working with Pythagoras and trigonometry:

- Investigating and understanding similarity, Pythagoras' Theorem and/or trigonometry
- Using and applying similarity, Pythagoras' Theorem and/or trigonometry to solve problems.

What the Australian Curriculum says about ‘Pythagoras and trigonometry’

Content descriptions

Strand | Measurement and geometry.

Sub-strand | Pythagoras and trigonometry.

Year 9 ◆ ◆ | ACMMG222
Students investigate Pythagoras’ Theorem and its application to solve simple problems involving right-angled triangles.

Year 9 ◆ ◆ | ACMMG223
Students use similarity to investigate the constancy of the sine, cosine and tangent ratios for a given angle in right-angled triangles.

Year 9 ◆ | ACMMG224
Students apply trigonometry to solve right-angled triangle problems.

Year level descriptions

Year 9 ◆ | Students use the trigonometric ratios for right-angle triangles.

Year 9 ◆ | Students apply ratio and scale factors to similar figures and solve problems involving right-angle trigonometry.

Year 9 ◆ | Students develop strategies in investigating similarity.

Achievement standards

Year 9 ◆ | Students interpret ratio and scale factors in similar figures.

Year 9 ◆ | Students explain similarity of triangles.

Year 9 ◆ | Students recognise the connections between similarity and the trigonometric ratios.

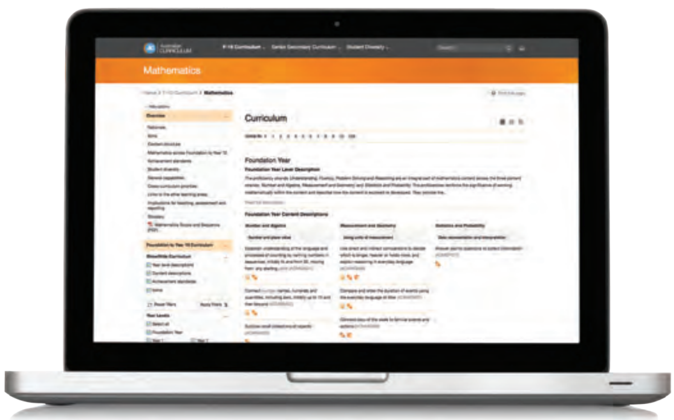
Year 9 ◆ | Students use Pythagoras’ Theorem to find unknown sides of right-angled triangles.

Year 9 ◆ | Students use trigonometry to find unknown sides of right-angled triangles.

Numeracy continuum

Using spatial reasoning

End of Year 10 ◆ | Students visualise, describe and analyse the way shapes and objects are combined and positioned in the environment for different purposes (Using spatial reasoning: Visualise 2D shapes and 3D objects).



Source: ACARA, Australian Curriculum: Mathematics, Version 8.1

Working with Pythagoras and trigonometry

Important things to notice about this sub-strand of the Australian Curriculum: Mathematics and numeracy continuum

What we are building on and leading towards in Year 9 'Pythagoras and trigonometry'

In Year 9 Pythagoras' Theorem and its applications are introduced. Students apply their understanding of similarity to establish the relationship between the corresponding sides of similar right-angle triangles. This is used to investigate the sine, cosine and tangent ratios in these triangles. The mathematical terminology 'opposite', 'adjacent' and 'hypotenuse' are introduced and students select and apply the appropriate trigonometric ratio to solve problems about angles and side lengths of right-angle triangles.

In Year 10 the terminology 'angles of elevation' and 'angles of depression' are introduced and students solve right-angle triangle problems applying a combination of Pythagoras' Theorem and trigonometry.

In Year 10A students solve problems in right-angle triangles in a three-dimensional context. Students graph trigonometric functions and establish their symmetrical and periodic properties. The use of sine and cosine rules for any triangle is also explored in 10A.

- **Pythagoras' Theorem and trigonometry provide contexts for developing and consolidating understanding of concepts within number** such as squares, square roots, angles, ratio and proportion, and statistics.
- **Pythagoras and trigonometry are important tools that students will choose when problem solving in measurement and geometry.** At this level we would want our students:
  - (having established Pythagoras' Theorem) to be able to recall the formula and the associated diagram
  - to recall appropriate labeling conventions
  - to recall a selection of square numbers to bring a certain level of automaticity to basic calculations
  - to know how to work out square and square root values on a calculator
  - to know that the theorem applies to all right-angled triangles
  - to know that the hypotenuse is the longest side
  - to know that the hypotenuse is diagonally opposite to the right-angle
  - to know that having any two side lengths of a right-angle triangle is sufficient to be able to calculate the third length, but for a right-angled isosceles triangle, knowing just the hypotenuse is sufficient to calculate the length of the other sides
  - to know that right-angle triangles can be identified within other polygons and circles
  - to know that Pythagoras' Theorem has many practical applications.
- **Conceptual understanding of Pythagoras and trigonometry must be built through concrete experiences.** As well as procedural fluency, the AC: Mathematics states that students should be both '*investigating* Pythagoras' Theorem and *applying* it' and this is an opportunity to show our students their capacity to look for patterns, to identify connections and establish rules in mathematics. In doing so we remove the need to 'tell' the students facts and instruct processes. Knowing that they have the capacity to create their own knowledge and understanding, rather than be a receiver and user of knowledge, underpins being a powerful learner.
- **Bridging from the concrete to the abstract.** Often we introduce trigonometry using similarity but then quickly expect students to use rules and procedures to calculate when applying the ratios. If we design the learning in this way, students may become fluent in applying trigonometry but are not likely to have conceptual understanding that it is really about identifying the equal ratios that exist in similar right-angled triangles. Hence it is best we ask students to explore these ideas fully before they are introduced to the formal mathematical language, algorithms and procedures.
- **Immediate success or conceptual understanding.** Teaching students a series of set processes for applying Pythagoras' Theorem and trigonometry to solve standard problems can yield some very immediate success in finding the correct answers. However without conceptual understanding, misconceptions become apparent when:
  - they become confused when they are required to learn about more than one ratio or need to find a side other than the hypotenuse
  - they try to record the solutions but use incorrect or inappropriate mathematical notation
  - they apply the rules to non right-angle triangles
  - they do not realise their answers are unreasonable, giving sin/cos ratios greater than 1 and accepting the hypotenuse not being the longest side
  - they can't apply the concept in an unfamiliar context, if the triangle is in a different orientation, or labelled unconventionally
  - they cannot retain their skills after they stop practising them daily.

  Without conceptual understanding, students will not be able to demonstrate reasoning, problem solving or even fluency.
- **Introducing formal mathematical proof.** Proving Pythagoras' Theorem in multiple ways is an opportunity to introduce students to more formal methods of proof. We should make a distinction between the conjectures that we make based on our measurements and these arguments which are *proving* rather than *verifying* the patterns we have observed.

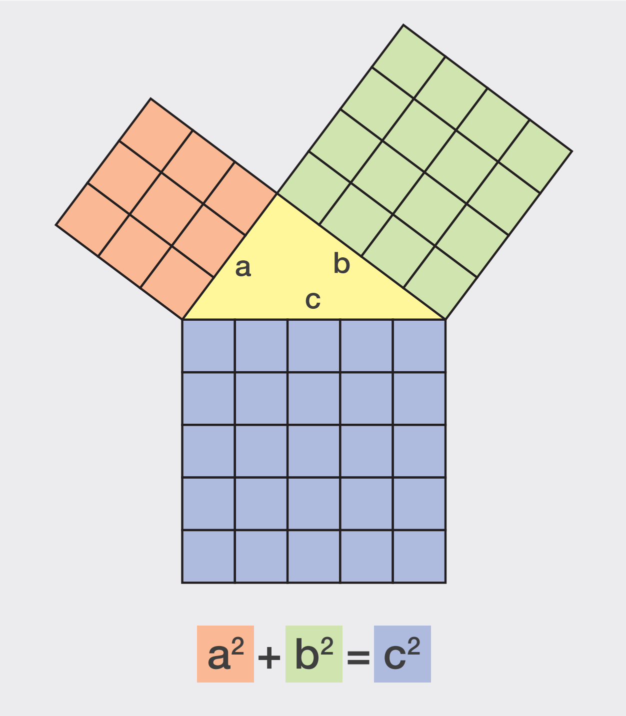
Engaging learners

Classroom techniques for teaching Pythagoras and trigonometry

Supporting learning with manipulatives

Idea six of Marilyn Burn's 10 Big Maths Ideas is to support learning with manipulatives and this is for the older students as well – http://www.scholastic.com/teachers/article/marilyn-burns-10-big-math-ideas.

It is possible to develop a conceptual understanding of Pythagoras' Theorem and trigonometry by drawing, cutting out, manipulating and measuring, both with concrete materials and digital objects. It enables all students access to this learning.



Real life contexts

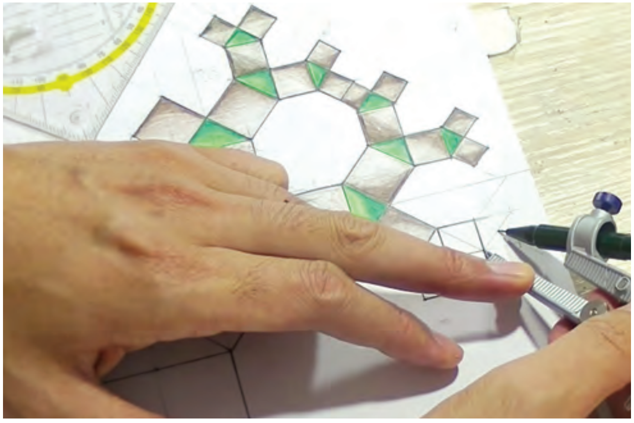
Our natural and built environments provide multiple practical and real life contexts for students to personalise, connect and apply their learning. Architectural applications and 'the knotted rope' example, demonstrate the power and simplicity of these geometrical concepts.

Pythagorean tree

For students who enjoy artistic and visual approaches to their learning, they will be engaged and intrigued by drawing a Pythagorean tree.

Watch the video at https://www.youtube.com/watch?v=9UtdjVWSluo, then ask students:

- *How many geometric principles can you see in this construction?*
- *Why is it a Pythagorean tree?*



Source: Dearing Wang (2014) *How To Draw Fractal Tree*, accessed at https://www.youtube.com/watch?v=9UtdjVWSluo

From tell to ask

Transforming tasks by modelling the construction of knowledge (Examples 1–13)

The idea that education must be about more than transmission of information that is appropriately recalled and applied, is no longer a matter for discussion. We know that in order to engage our students and to support them to develop the skills required for success in their life and work, we can no longer rely on a 'stand and deliver' model of education. It has long been accepted that education through transmission of information has not worked for many of our students. Having said this, our classrooms do not necessarily need to change beyond recognition. One simple, but highly effective strategy for innovation in our classrooms involves asking ourselves the question:

What information do I need to tell my students and what could I challenge and support them to develop an understanding of for themselves?

For example, no amount of reasoning will lead my students to **create the name for sine or cosine ratios** for themselves. They need to receive this information in some way. However, it is possible my students can be challenged with questions that will result in them identifying **constant ratios for right-angled triangles**, so I don't need to instruct that information.

At this stage of development, students can develop an understanding of Pythagoras and trigonometry through conducting their own experiments. When teachers provide opportunities for students to predict, identify, describe and represent the outcomes of the experiments and compare them to theoretical expectations, they require their students to generalise. Telling students the laws of trigonometry removes this natural opportunity for students to make conjectures and verify connections that they notice.

When we are feeling 'time poor' it's tempting to believe that it will be quicker to tell our students a formula, rather than ask a question (or series of questions) and support them to establish a formula for themselves. Whether this is true or not really depends on what we have established as our goal. If our goal is to have students recall and apply a particular formula during the current unit of work, then it probably is quicker to tell them the formula and demonstrate how to apply it. However, when our goal extends to wanting students to develop conceptual understanding, to learn to think mathematically, to have a self-concept as a confident and competent *creator* and *user* of mathematics, then telling students the formulae is a false economy of time.

When we challenge our students to *establish* theorems, we model that algebra can be powerful and useful. We provide our students with an authentic context for working algebraically. Telling students formulae removes this opportunity for students to generalise.

Curriculum and pedagogy links

The following icons are used in each example:

The '**AC**' icon indicates the Australian Curriculum: Mathematics content description(s) addressed in that example.

The '**Bringing it to Life (BitL)**' tool icon indicates the use of questions from the *Leading Learning: Making the Australian Curriculum Work for Us* resource.

The Bringing it to Life tool is a questioning tool that supports teachers to enact the AC: Mathematics Proficiencies: http://www.acleadersresource.sa.edu.au/index.php?page=bringing_it_to_life

The '**From tell to ask**' icon indicates a statement that explains the transformation that is intended by using the task in that example.

This idea of moving 'From tell to ask' is further elaborated (for Mathematics and other Australian Curriculum learning areas) in the 'Transforming Tasks' module on the *Leading Learning: Making the Australian Curriculum work for Us* resource: http://www.acleadersresource.sa.edu.au/index.php?page=into_the_classroom

Look out for the purple pedagogy boxes, that link back to the SA TfEL Framework.

From tell to ask examples	
Example 1: Establishing the hypotenuse is the longest side Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.	ACMMG222 ◆ ◆
Example 2: Establishing Pythagoras' Theorem Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.	ACMMG222 ◆ ◆
Example 3: Pythagorean investigation Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.	ACMMG222 ◆ ◆
Example 4: Establishing Pythagorean triads Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.	ACMMG222 ◆ ◆
Example 5: A piece of knotted rope – a historical application of Pythagoras' Theorem Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.	ACMMG222 ◆ ◆
Example 6: Why can't I …? – tackling misconceptions Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.	ACMMG222 ◆ ◆
Example 7: Constructing a volleyball court – practical application of Pythagoras' Theorem Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.	ACMMG222 ◆ ◆
Example 8: A special ratio – introducing the sine ratio through similarity Students use similarity to investigate the constancy of the sine, cosine and tangent ratios for a given angle in right-angled triangles.	ACMMG223 ◆ ◆
Example 9: Connecting student understanding to mathematical protocols Students apply trigonometry to solve right-angled triangle problems.	ACMMG224 ◆
Example 10: What about the triangles we rejected? – sine ratio of any angle Students use similarity to investigate the constancy of the sine, cosine and tangent ratios for a given angle in right-angled triangles.	ACMMG223 ◆ ◆
Example 11: What if we don't know the side opposite? – introducing the cosine ratio Students use similarity to investigate the constancy of the sine, cosine and tangent ratios for a given angle in right-angled triangles.	ACMMG223 ◆ ◆
Example 12: Set a problem for someone else Students apply trigonometry to solve right-angled triangle problems.	ACMMG224 ◆
Example 13: Pythagoras' short problems Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles. Students apply trigonometry to solve right-angled triangle problems.	ACMMG222 ◆ ◆ ACMMG224 ◆

Example 1: Establishing that the hypotenuse is the longest side

ACMMG222 ◆ ◆

Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/ relationships can you see?

Can you represent/calculate in different ways?

Reasoning proficiency:

In what ways can your thinking be generalised?

What can you infer?

Instead of ***telling*** students about the properties of right-angled triangles, we can challenge students to recognise the relationships for themselves, by ***asking*** questions.

Students learn by noticing: Invite their curiosity

Rather than stating that the longest side in a triangle is always opposite the largest angle, ask students to notice this relationship for themselves.

Figure 1

Ask students:

- *Looking at Figure 1:*
  - *Can you see any connections between the position of the largest angle and the largest side of a triangle?*
  - *Can you see any connections between the position of the smallest angle and the smallest side?*
- *Is there a rule that we could always use to describe the relationships between the position of the largest side and the largest angle of a triangle? Why is this the case?*
- *Is this true for all polygons (quadrilaterals, pentagons, etc)?*
- *What's the best way to show your thinking?*
- *Can you convince someone who isn't sure?*

Example 2: Establishing Pythagoras' Theorem

ACMMG222 ◆ ◆

Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/ relationships can you see?

Can you represent/calculate in different ways?

Reasoning proficiency:

In what ways can your thinking be generalised?

What can you infer?

Instead of ***telling*** students about Pythagoras' Theorem, we can challenge students to recognise the relationships between the side lengths in right-angled triangles for themselves, by ***asking*** questions.

In this activity, instead of telling students about Pythagoras' Theorem, we can challenge students to recognise the relationships between the side lengths in right-angled triangles for themselves.

Ask students:

- *What shapes can you see in the pattern in Figure 2a?*
- *What triangles can you see?*
- *What's the same about all of the triangles that you can see?* (All right-angled, but how do you know?)
- *What's different about the triangles that you can see?* (Different sizes and orientations.)

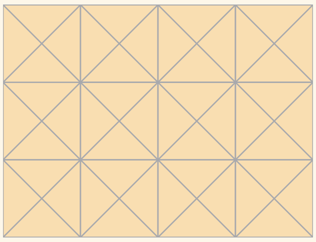


Figure 2a

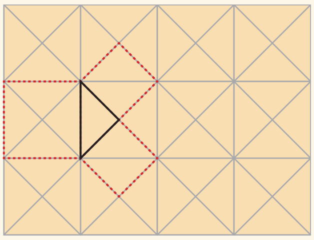


Figure 2b

Ask students to choose a triangle in the shape and for each of the 3 sides of that triangle, look for a square that is constructed on it (see Figure 2b). Then ask students:

- *How are the sizes of the squares that surround a triangle connected?*
- *Where is the largest square located on the triangle?*
- *Is the largest square always in the same position on the triangle?*
- *Is there a rule that you could always use to describe the relationship between the sizes of the squares that surround the triangles?*
- *What type of triangles have we been working with?* (Isosceles right-angle triangles.)
  - *What if the triangles were not right-angled?*
  - *Does this rule work for all triangles? What do you think? Convince me/yourself!*
  - *What if the triangles were not isosceles right-angle triangles, does that rule work for all right-angle triangles (ie for right-angled scalene triangles)? What do you think? Convince me/yourself!*
  - *How might you express the relationship algebraically?*

Activating reasoning

Asking 'Why?' after any of the 'What if …?' questions, activates students communication and allows the teacher to observe and develop logical sequencing of thoughts, and correct use of terminology in their students' explanations.

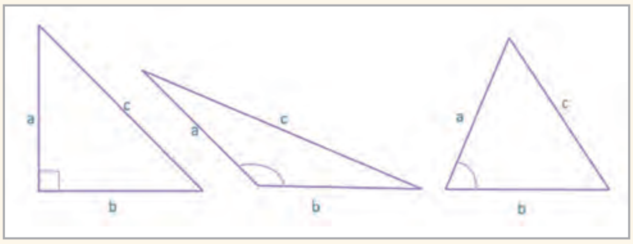


Figure 3a   Figure 3b   Figure 3c

Show students Figure 3a and ask:

- *What if we start with a right-angle triangle and we 'open out' the sides that form the right angle? Now, how does c^2 relate to $a^2 + b^2$? Why?* (See Figure 3b.)
- *What if we start with a right-angle triangle and we 'close up' the sides that form the right angle? Now, how does c^2 relate to $a^2 + b^2$? Why?* (See Figure 3c.)

To have a manipulative with an elastic side (c), makes this experience more authentic and students are more likely to visualise this as a prompt to check that the triangle is right-angled before applying $a^2 + b^2 = c^2$.

Example 3: Pythagorean investigation

ACMMG222 ◆ ◆
Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/relationships can you see?

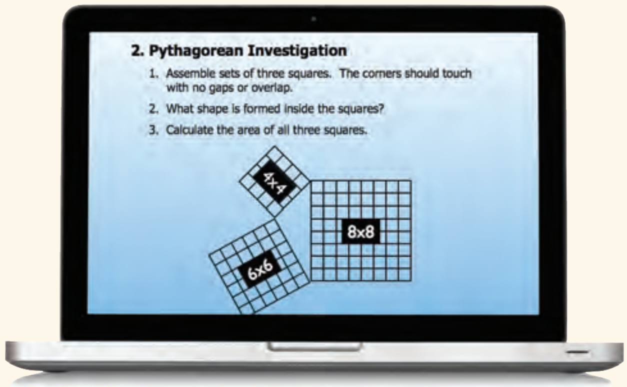
Can you represent/calculate in different ways?

Reasoning proficiency:

In what ways can your thinking be generalised?

What can you infer?

Instead of ***telling*** students about Pythagorean triads (triplets), we can challenge students to construct them in a practical way for themselves, by ***asking*** questions.



This activity is from Dan Meyer's blog: dy/dan. The **Pythagorean investigation** can be accessed at: http://blog.mrmeyer.com/2007/geometry-day-65-pythagorean-theorem/

In this activity students use a range of squares (eg 1×1, 2×2, 3×3, on up to 15×15) to build triangles out of the sides. They will need to write down the areas of the three squares and whether the triangle formed is obtuse, acute, or right.

This activity links squares of the sides to the triangle. It also reinforces the triangle inequality in that the sum of the lengths of the two smaller sides must exceed the third for it to be possible to construct a triangle.

When students are investigating with particular cases, ask them to predict what they think will happen, before they use the manipulatives. Ask students:

- *What pattern do you see? How could you check that?*
- *Which three squares are you using next? What do you expect will happen? Ok, let's see if you're right.*
- *What do you think now? Does that make sense?*
- *What connections do you notice for the triangles that are right-angled?*

Check a maths dictionary and see what a 'Pythagorean triad' is, then ask students:

- *Why are they called that? Is there another name? Why are these the odd ones out?*
- *Are the triads all the same? Why might we group 3, 4, 5 and 6, 8, 10? Are there any others like this?*
- *Can you write an explanation for a Year 8 student to explain what a Pythagorean triad is?*

We need to encourage students to consider when the connections will, or will not exist so they will apply the theorem appropriately when they are solving problems. Encourage them to think more deeply about their exploration, make conjectures and ask themselves if it makes sense.

Backwards questions

A 'backwards' question requires a little more thinking about than a standard application question. We ask this type of question to challenge students to work flexibly with a concept. There are many ways that teachers can do this. Two possible options include asking questions where students need to complete missing information and asking questions that start by giving the students an answer.

The 4x4 and 6x6 did not make a right-angled triangle with the 8x8. Ask students:

If we wanted to make a right-angle triangle using two of these squares:

- *What could you try? Would you need to make the 4x4 bigger or smaller? Why?*
- *How could you work out what it has to be? What is another way?*
- *What if you had to use the 4x4 and 6x6 squares?*

Some students will use the manipulatives, possibly drawing a square on graph paper that fits and measure the sides. Others may use trial and error to find the number such that $36 + (?)^2 = 64$; others may use algebraic thinking to solve the problem. Reward all methods for good thinking. We can use this as a teaching moment as students share their thinking in class discussion and compare the accuracy, ease and elegance of the methods. This is also an opportunity to discuss how the degree of accuracy required is dependent on context. Often the trial and error method is accurate to one decimal place, which is sufficient for laying a water pipe but not for building a bridge.

Example 4: Establishing Pythagorean triads

ACMMG222 ◆ ◆

Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/ relationships can you see?

Can you represent/calculate in different ways?

Reasoning proficiency:

In what ways can your thinking be generalised?

What can you infer?

Instead of ***telling*** students about Pythagorean triads (triplets) and their multiples, we can challenge students to discover them for themselves, by ***asking*** questions.

We can challenge students to identify that multiples of Pythagorean triplets are also Pythagorean triplets by showing them Figure 4, then asking:

- *Are these triangles right-angled? How do you know? Convince me.*

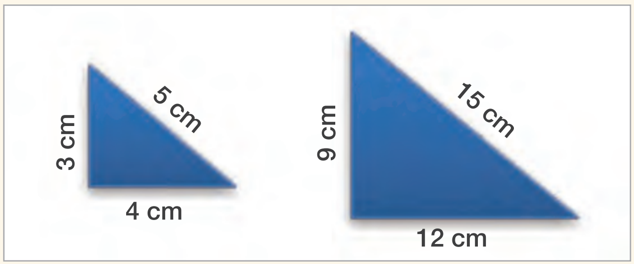


Figure 4

Constructing knowledge: From information to understanding

In this questioning sequence we have moved from noticing a possible pattern, through to using that information to suggest another possibility; checking that possibility, then generalising the relationship, and finally into a problem solving task about working out other Pythagorean triads.

We could introduce this idea by first stating:

These right-angle triangles are special because the length of all three sides is a whole number.

Then asking questions such as:

- *Can you see any connections between these triangles?*
- *Now that you have noticed a relationship, could you suggest another right-angle triangle with all three sides of a whole number value?*
- *How could you check that you have produced another right-angle triangle?*
- *Is there a rule that you could use to describe how you worked out another possibility?*
- *Is there another triangle that you could produce using your rule? How many could you make?*

Now that students have identified a relationship between the lengths of the sides of the triangles, inform them that these special triangles are called Pythagorean triads (or Pythagorean triples). Then ask:

- *Is it possible to have a Pythagorean triplet that is not in this ratio (3:4:5)? Convince me.* (This question models to students that when they see a possible connection, they should try to extend their thinking to other cases to see if the connection continues.)

In relation to Pythagoras' Theorem, given that students first learn to calculate the hypotenuse using the perpendicular sides; a backwards question would be one in which the hypotenuse is one of the given pieces of information.

Backwards questions

A 'backwards' question requires a little more thinking about than a standard application question. We ask this type of question to challenge students to work flexibly with a concept. There are many ways that teachers can do this. Two possible options include asking questions where students need to complete missing information, and asking questions that start by giving the students an answer.

Students need to be flexible in their application of Pythagoras' Theorem. Show students the steps in Figure 5 (on page 12) and ask:

- *If the longest side of a right-angle isosceles triangle is 20 cm, what are the lengths of the other two sides? What if the right-angle triangle is not isosceles, what are some other possibilities for the lengths of the perpendicular sides?* (Because it is an isosceles triangle, the length for the other two sides can be determined. As well as algebraic methods, ie solving $x^2 + x^2 = 20^2$, also consider a 'construct and measure' method as a lower entry point or to verify the algebraic solution.)

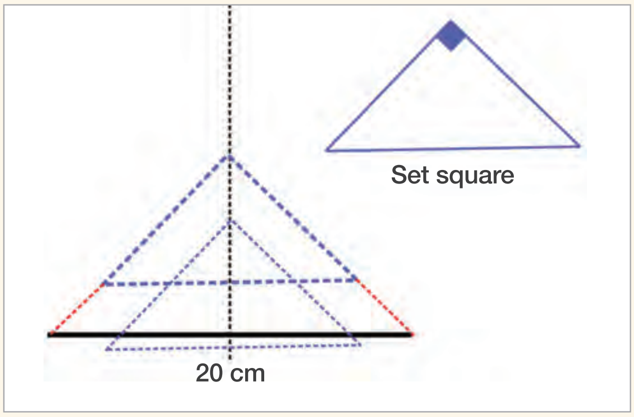


Figure 5

Draw a line of 20 cm and its perpendicular bisector on graph paper. Place the set square so that the vertex of the right-angle is centred on this line. Move it up and down until the extension of the legs of set square, complete a right-angle isosceles triangle and measure the lengths of its sides.

- *The three sides of a triangle are 3 cm, 5 cm and 6 cm. Is this a right-angle triangle?*
- *Two sides of a triangle are 12 cm and 13 cm. This is a right-angle triangle. What's the length of the third side? Is there another possibility?*

The last question may become a problem solving question for many students, so teachers can support them to make progress (but keep the thinking with the student) by asking questions such as:

- *Would a diagram help?*
- *Where else could the values 12 and 13 be placed?*
- *Why can't 12 be the hypotenuse?*

Example 5: A piece of knotted rope – a historical application of Pythagoras' Theorem

ACMMG222

Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/ relationships can you see?

Can you represent/calculate in different ways?

Reasoning proficiency:

In what ways can your thinking be generalised?

What can you infer?

Instead of ***telling*** students about how the knotted rope was used, we can challenge students to apply Pythagoras' Theorem in practical ways, by ***asking*** questions.

The knotted rope was used widely by architects as far back as 1180AD, over 800 years ago. A knotted rope has at least 13 knots equally spaced along the rope. Architects used the rope to construct shapes such as an equilateral triangle, but it was most essential for making sure that the angles in their buildings were right-angles.

Ask students to form small groups and give each group a different length of rope and a tape measure. This can be an outdoor task using longer sections of rope. Have them create their own knotted rope and use it to construct both an equilateral triangle and a right-angled triangle, then ask:

- *What have you learnt about the lengths of the sides of right-angled triangles? How could you use this knowledge to construct a right-angle?*

Students can easily construct an equilateral triangle by holding the knots that make all the sides the same length.

A right-angled triangle will be created when students hold the knots that give sides of 3, 4 and 5. This is actually using the converse of Pythagoras' Theorem that if $a^2 + b^2 = c^2$ then the triangle must be right-angled. Ask students:

- *How could we check the accuracy of our shapes?*
- *Are all our triangles the same?*

The students had been given different lengths of rope so while the angles will be the same, the sides will be different lengths, ie they are similar triangles. If the actual length of the sides of the triangles is measured, the students' measurements will all be multiples of the 3, 4, 5 triad. Complete the activity by asking students:

- *How else could you use the rope?*

Example 6: Why can't I ...? – tackling misconceptions

ACMMG222 ◆ ◆

Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.

Questions from the BitL tool

Understanding proficiency:

Can you represent/calculate in different ways?

Reasoning proficiency:

In what ways can you communicate?

What can you infer?

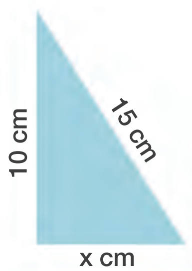
Instead of ***telling*** students about the mistakes students might make, we can ask them to identify mistakes in work samples by ***asking*** questions.

Tackling misconceptions: Knowing what it isn't

We can challenge students' reasoning and address common misunderstandings by asking *'Why can't I …'*; *'Why is it not …' and 'What's wrong with …'* questions. Students can be given problems from fictional students which are answered incorrectly, without working out. They can be asked to give feedback, summarising the mistakes that the students have made and how to avoid that mistake in the future.

Show students an example of a work sample which was answered incorrectly, like the one in Figure 6, then ask:

- *Is this answer correct?* (Notice the student has not identified the hypotenuse correctly and has also assumed that x is the positive square root of 325 without comment. While the negative solution is not relevant as x is the length of a side, this should be acknowledged with a comment such as '$x > 0$'.)



$x^2 = 10^2 + 15^2$

$x^2 = 100 + 225$

$x^2 = 325$

$x = \sqrt{325}$

$x = 18.03$ cm (2 d.p.)

← *Is this answer correct?*

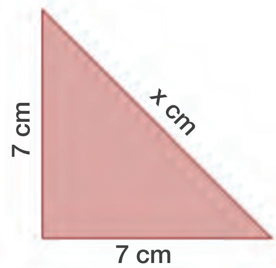
Figure 6

Ideally there should be a mixture of correctly and incorrectly answered questions for students to give feedback on.

Teachers can also use this type of activity to draw attention to the importance of students communicating their thinking/showing how they achieved their answer. When students are trying to give feedback on questions where they cannot 'see' the calculations, they can begin to realise the value of showing the thinking. Ask students:

- *Do you think that this student knows how to use Pythagoras' Theorem? Can you be sure? Why? Why not? What would help you to decide?*

In Figure 7, the solution 10 cm is correct if the value has been rounded to the nearest whole number, but as the student hasn't indicated this in his/her solution are you confident about their understanding?



Answer
10 cm

Figure 7

Show students Figures 8a–8d (see page 14), and ask:

- *What's wrong with …?* (**Figure 8a** is not a right-angled triangle and so Pythagoras' Theorem does not apply. **Figure 8b** is not a feasible triangle as the hypotenuse is not the largest side. In **Figure 8c**, the side labeled 'a' is the hypotenuse, so Pythagoras' Theorem should be stated as $a^2 = b^2 + c^2$. In **Figure 8d**, much of the reasoning has not been provided, which has led to an error in determining the expression for calculating 'a'.)

If these examples are displayed on a board in the classroom, students could add examples of possible misconceptions, annotated with explanations and corrections. Presenting students with multiple-choice questions in which the options contain common misconceptions can reveal useful information for the teacher. Asking students to communicate why they made their selection, can offer further insight into the students' thinking.

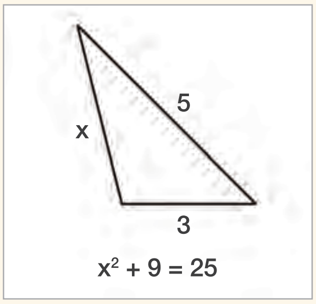


Figure 8a

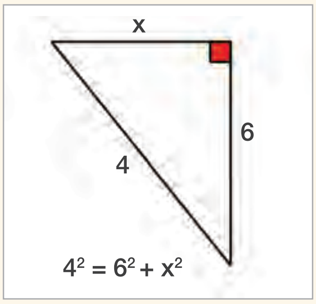


Figure 8b

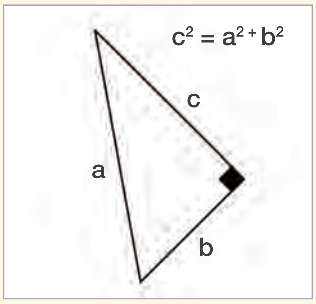


Figure 8c

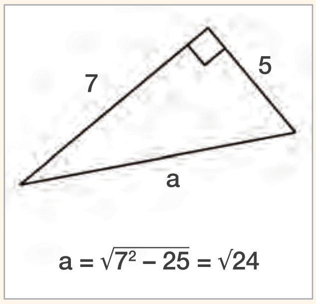


Figure 8d

Example 7: Constructing a volleyball court – practical application of Pythagoras' Theorem

ACMMG222 ◆ ◆

Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.

Questions from the BitL tool

Understanding proficiency:

Can you represent/calculate in different ways?

Reasoning proficiency:

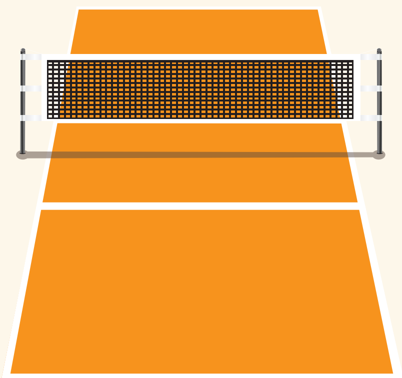
In what ways can you communicate?

Instead of ***telling*** students to apply Pythagoras' Theorem to complete a task, we can challenge students to identify this for themselves, by ***asking*** questions.

It is important to remember that solving word problems or problems from images, is NOT necessarily problem solving. One element of fluency is choosing and using mathematics flexibly. If the student has knowledge of Pythagoras' Theorem, then applying that knowledge, even within a context, is expected. Becoming familiar and comfortable with the application of Pythagoras' Theorem is useful, but it's important that teachers do not mistake this for problem solving.

Present the following practical application problem to students:

You have one long tape measure, 4 pegs that can be pushed into the ground, 7 metres of string, and some chalk, grass paint or tape. Create a perfect rectangle of length 5 metres and width 7 metres as the sidelines of a volleyball court.



Students could use Pythagoras' Theorem to calculate the diagonals and use this measurement with sides of 7 m and 5 m, in the construction of the rectangle. Alternatively they might use the string and a peg as a makeshift compass and scribe arcs to create right-angles. They may also trust their judgement by eye and use neither of these techniques. There will be some other creative solutions too. If the students thought to use Pythagoras' Theorem in the first instance then the teacher has seen that they have chosen to apply their understanding.

If they didn't choose Pythagoras' Theorem to assist with the construction, they could be asked to use it to prove that their court is actually a rectangle and not just a parallelogram.

Value diverse thinking: Utilise peer tuition

All processes can be valued, even though they use different levels of mathematics. All students benefit from seeing different ways to achieve the same goal. Being able to visualise a rectangle and estimate lengths and angles is a valuable skill, but sometimes more accuracy is required, so we need a more precise technique. This is a perfect opportunity for peer tutoring to be used.

Example 8: A special ratio – introducing to the sine ratio through similiarity

ACMMG223 ◆ ◆

Students use similarity to investigate the constancy of the sine, cosine and tangent ratios for a given angle in right-angled triangles.

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/ relationships can you see?

Reasoning proficiency:

In what ways can you communicate?

What can you infer?

Instead of ***telling*** students about the trigonometric ratios we can challenge students to identify them for themselves, by ***asking*** questions.

Create a class set of similar triangles which are right-angled and have an angle of 30°. Ask all students to construct a right-angled triangle with a 30° angle of any size. Cut them out and have students compare their triangles. Have a large triangle that students can also use for comparison, in particular to see if the 30° angle has been drawn correctly. Then ask:

- *How is your triangle the same or different to other students' triangles or to the large triangle? What mathematical word describes the relationship between all our triangles?* (The class will have created similar triangles if drawn accurately.)
- *What have we learnt are the properties of similar triangles? How could we check if your triangle was similar to the large triangle?* (Students have probably checked angles by superimposing them on each other. If the angles are the same, the triangles will be similar. In fact they only have to check that two of the angles are the same as the third must be the same for them all to add up to 180°.)

Optional: If students have already studied similar triangles they would know that the sides must all be in the same ratio. To check the ratio of the sides, students can divide each of the side lengths of the large triangle by their side lengths for corresponding sides, and all three ratios they calculate should be the same. Ask students:

- *Does everyone get the same number (ratio) for each of their three sides? Why not?* (There will be some variation due to the inaccuracies in drawing the triangles as well as in measuring the lengths.)
- *Does everyone get the same number (ratio) as each other? Why not?*
- *If the number (ratio) you get is 2, what does that mean about our triangles?* (The large triangle's sides are twice as big as yours, or the large triangle is what yours would look like after it has been enlarged by a scale factor of 2.)

Introducing the concept of the sine ratio as a special ratio (or connection) between the side opposite and the hypotenuse of the right-angled triangle.

State that you have noticed an interesting relationship between two of the sides of the large triangle, and you want everyone to check if it is the same for their triangle.

If it has not become apparent before this, discuss that the class needs a way to talk about the sides of the triangle. Mathematicians already have agreed on some names and ways to label triangles.

Show students Figure 9, then ask:

- *Which side might be the Hypotenuse? Which side might be referred to as Opposite to A?*
- *Which side might be referred to as Adjacent (next to) A? There are two sides 'next to' angle A. How will you know which one is the 'adjacent' side?*

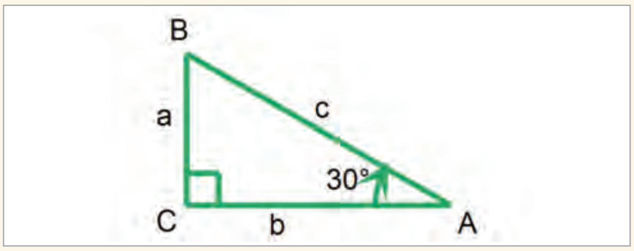


Figure 9

Notice the labeling protocol to label the angle and side opposite with the same letter. Ask students to label their triangles (Opposite 30°, Adjacent to 30° and Hypotenuse) and check the labels with their partner. Ask them to measure lengths opposite 30°, the hypotenuse, and record their results on the board. Students that finish first can design the table to record the information (see Figure 10 on page 16), or measure the teacher's large triangle.

You can also set a challenge to say that someone said that 'bigger triangles are more accurate', then ask students:

- *Do you agree or disagree? How could you check?* (Errors due to inaccuracies in measurement are less significant for bigger measurements as it is a smaller percentage error.)

- *What pattern do you notice? (Make a conjecture.)* (Students often notice that the hypotenuse is double the side opposite; allowing for measurement inaccuracies.)
- *What would you expect the answer to be if we worked out the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$?* (Add the extra column and determine if the ratios are all approximately ½.)

This is the special ratio of $\frac{\text{opposite}}{\text{hypotenuse}}$ for an angle of 30° in a right-angled triangle.

Triangle belongs to:	Length of side opposite (a)	Length of hypotenuse (c)	Ratio $\frac{\text{opposite}}{\text{hypotenuse}}$
Teacher			
Student 1			
Student 2			
Student 3			

Figure 10

Use a collection of triangles, with at least one which does not have a right angle and some that do not have 30°. (Note: these these will also be used in Example 10 on page 18.) Ask students:

- *From the collection of right-angled triangles, how could you identify which have an angle of 30° without using a protractor? Is there another way?*
- *Measure the hypotenuse for one of the selected triangles (right-angled with 30°). Can you work out what the side opposite would be without measuring it? Measure the side opposite to verify.*
- *Measure the opposite side for one of the selected triangles. Can you work out what the hypotenuse would be? Measure the hypotenuse to verify your answer.*
- *Can you write to a Year 8 student explaining about the special ratio you discovered today? Explain how you might use this new understanding to find a side length or an angle you don't know. Are there any limitations to when you can apply this new knowledge?*

The special ratio [$\frac{\text{opposite}}{\text{hypotenuse}}$] is called the sine ratio and mathematicians write $\sin 30° = \frac{1}{2}$.

Verify this using a calculator and ask:

- *Can you predict what the calculator will show as the sin 29°? Why? How do the sides change when the angle changes? Check.*
- *Can you predict what the calculator will show for 31°? Why? How do the sides change when the angle changes? Check.*
- *What do you wonder? Check any other angles that interest you.*

Angles of 89° and 1° are interesting because they are extreme. Non-integer values such as 29.5° and 30.1° can be checked. Some curious students may try values greater than 90° using their calculator, which links to trigonometry and the unit circle which is part of the Year 10A course.

- *Why do you think I asked you to draw right-angled triangles with a 30° angle? Has that triangle got special properties that other right-angled triangles don't have?* (We need to establish that 30° is not the only angle that has sine (special) ratio but that it was chosen because the value was ½ which made it easier to see the pattern.)

Constructing knowledge: From information to understanding

In this questioning sequence we have moved from noticing a possible pattern, through to using that information to suggest other possibilities, checking the possibility, then generalising the relationship.

Example 9: Connecting student understanding to mathematical protocols

ACMMG224 ◆
Students apply trigonometry to solve right-angled triangle problems.

Questions from the BitL tool
Understanding proficiency:
Can you represent/calculate in different ways?
Reasoning proficiency:
In what ways can you communicate?

Instead of ***telling*** students to record processes using formal mathematical protocols when solving problems using trigonometry, we can challenge students to identify them for themselves, by ***asking*** questions.

By this stage, students are using the concepts of trigonometry (from the Greek: triangle measure) and they have been giving reasons for their thinking, but they have not been using the written protocols that mathematicians use to communicate and are not likely to develop these for themselves.

Mathematical protocols and representations: Linked to conceptual understanding

Whenever students develop the understanding themselves, there will come a stage when they need to be made aware of the way other mathematicians record their thinking. As teachers, we must facilitate a transition period where a student's current understanding is connected to mathematical protocols in a blended fashion. If this is a sudden and irreversible process, it creates a disjunction where the benefit of students creating their own knowledge will be lost. Facilitate the transitioning by requiring all students to write their reasoning in words when they solve a problem. They often find this difficult, repetitive, and/or time consuming and so are generally responsive to learning an easier method. The most important aspect of the transition is linking what they have been saying to the interpretation of the mathematical solution.

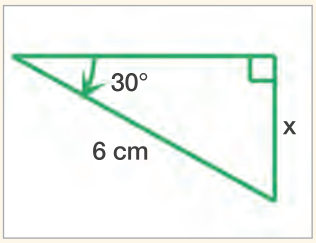


Figure 11

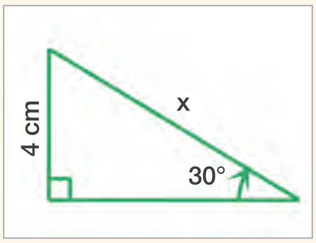


Figure 12

Show students Figure 11 and ask:

- *What would the value of x be in this triangle? x = ...*
- *Write your reasoning in words.* (An example of reasoning could be: The special ratio for the opposite divided by the hypotenuse for a 30° angle in a right-angled triangle is half. Therefore as x is the side opposite and 6 is the hypotenuse, x needs to be half of 6, the hypotenuse, so it is 3.)
- *Did you have the same information? Is there anything missing?*

Look at the mathematical working and ask students:

- *We have solved a lot of problems using this reasoning, could this be written in a shorter more concise way?*
- *Can you link what you wrote in words to the mathematical language?* Mathematicians would set out the problem you just did like this:

The special ratio for the opposite divided by the hypotenuse for a 30° angle in a right-angled triangle is half. x is the side opposite and 6 is the hypotenuse.	Sin 30° = $\frac{x}{6}$
x needs to be half of 6, 6 x ½ , or you can use a calculator.	6 x sin 30° = x
So it is 3.	x = 3

- *What are the advantages and disadvantages of the two methods?*

Students can practise solving a range of problems, showing their reasoning using the methods. Present another problem using Figure 12 and ask students:

- *What would the value of x be in this triangle? x = ...*
- *How is this problem the same or different to the one we just did?*
- *How might you solve this problem? Explain your reasoning and write it in words.*
- *Could you write a mathematical solution similar to the one we used before? Does it link to the reasoning we write in words to explain our thinking?*

This problem you have completed can be easily worked out in your head, but when they get more difficult you need a way to record and calculate using what you know about solving equations:

$\sin 30° = \frac{4}{x}$

$x \times \sin 30° = 4$

$x = \frac{4}{\sin 30°}$ (4 ÷ ½ = 8 or you can use a calculator)

$= 8$

Practise recording some of the calculations.

Example 10: What about the triangles we rejected? – sine ratios of angles other than 30°

ACMMG223 ◆ ◆

Students use similarity to investigate the constancy of the sine, cosine and tangent ratios for a given angle in right-angled triangles.

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/ relationships can you see?

Can you answer backwards questions?

Reasoning proficiency:

In what ways can you communicate?

What can you infer?

Instead of ***telling*** students about the inverse trigonometric functions on the calculator, we can challenge students to identify them for themselves, by ***asking*** questions.

This activity uses the right-angled triangles that are not 30° from *Example 8: A special ratio – introducing to the sine ratio through similiarity* on page 15. Pick one of the triangles, like the one in Figure 13, and complete a table like the following:

	Length of side opposite (a)	**Length of hypotenuse (c)**	**Ratio** $\frac{\text{opposite}}{\text{hypotenuse}}$
Triangle 5	1.71	5	0.342

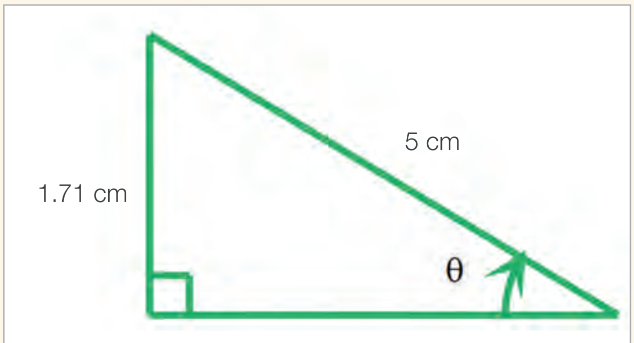


Figure 13

Ask students:

- *What do you notice? Why is this different? Do you think the angle would be bigger or smaller than 30°? Why?* (The sine ratio is not ½ because the angle was not 30˚.)
- *How could you find out what the angle was with your calculator?* (Using trial and error with a calculator, find the angle that has a sine value of $\frac{1.71}{5} = 0.342$:
  Sin 25 ≈ 0.423 Sin 22 ≈ 0.375 Sin 20 ≈ 0.342)

Let students know that a calculator can find this angle for them. Ask them to find the ‘sin^{-1}’ button, then ask:

- *Where was it? Why do you think it is there? What do you think it does? How could we check this?* (The inverse sine button (sin^{-1}) is above the sine button, just as the √ button is above the x^2 button, because they are inverse functions. So $\sin^{-1}(\frac{1.71}{5}) \approx 20°$.)

Consider **sin^{-1} (½) = ?** and ask students:

- *What do we expect? Check.*

Encourage students to make the link between the calculator, mathematical notation and the language we have been using:

- Write ‘The angle whose (special ratio) sine ratio is’ ½ is 30˚. This phrase, ‘The angle whose (special ratio) sine ratio is’, can be used whenever writing ‘sin^{-1}’ to consolidate the meaning of the mathematical notation.
- Verify using the calculator sin^{-1} (½) = 30˚.
- By measuring sides and then using the inverse sine function (sin^{-1}), determine the angles of the other triangles we have rejected. Verify your calculations by measuring the angle with a protractor.

Ask students:

- *How is this problem the same or different to the one we did in Example 9?*
- *Could you write a mathematical solution similar to the ones we used before? Does it link to the reasoning we write in words to explain our thinking?*

Mathematicians would set your answer out like this:

$\sin\theta = \frac{1.71}{5}$	The special ratio (sine) for this unknown angle is opposite over hypotenuse which is
$\theta = \sin^{-1}(\frac{1.71}{5})$	The unknown angle can be found with the calculator, using the inverse sine function of $\frac{1.71}{5}$ ie the angle whose (special ratio) sine ratio is $\frac{1.71}{5}$
$\theta \approx 20°$	The unknown angle is about 20 degrees. I could check this with the calculator by finding sin20 and the value of 1.71 ÷ 5 = 0.342

Mathematical protocols and representations: Linked to conceptual understanding

Whenever students develop the understanding themselves, there will come a stage when they need to be made aware of the way other mathematicians record their thinking. As teachers, we must facilitate a transition period where a student's current understanding is connected to mathematical protocols in a blended fashion. If this is a sudden and irreversible process, it creates a disjunction where the benefit of students creating their own knowledge will be lost.

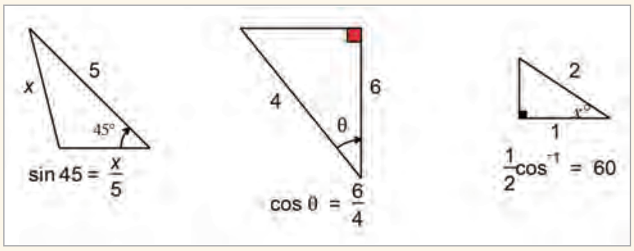


Figure 14

Tackle common misconceptions by asking students to consider work samples from a fictitious student, like the ones in Figure 14, and give the student feedback on their reasoning.

Reasoning might include:

- The first is not a right-angled triangle and so right-angled trigonometry does not apply.
- The second is not a feasible triangle as the hypotenuse is not the largest side.
- In the third example, the mathematical notation is meaningless. Having $\cos^{-1}$ without a number is like writing √ on its own. 'The square root of what?' $\cos^{-1}$ 'The angle whose cos is what?'
  Cos x = ½
  x = $\cos^{-1}$(½)
  x = 60°

Example 11: What if we don't know the side opposite? – introducing the cosine ratio

ACMMG223 ◆ ◆
Students use similarity to investigate the constancy of the sine, cosine and tangent ratios for a given angle in right-angled triangles.

Questions from the BitL tool
Understanding proficiency:
What patterns/connections/relationships can you see?
Reasoning proficiency:
In what ways can you communicate?
What can you infer?

Instead of ***telling*** students about the cosine and tangent ratios, we can challenge students to identify them for themselves, by ***asking*** questions.

Now that students have knowledge of Pythagoras' Theorem, the sine ratio and scale diagrams; they may use these ideas or a combination to solve the following problem.

Do not give hints before they start the problem. If students use the sine ratio incorrectly, they will be able to check with other groups to discuss why their answers are different and can then self-correct.

Show students the following problem in Figure 15:

Problem: The slippery dip in a children's playground has dimensions as shown.

Figure 15

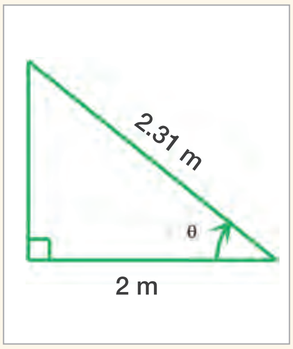


Figure 16

For safety reasons, the angle the slippery dip makes with the ground cannot be greater than 35°. Ask students:

- *Is the slippery dip 'safe'?*

This problem can be represented by the diagram in Figure 16. However, the triangle drawn over the slippery dip does not appear right-angled. When a representation of a real situation is used in problem solving, it's important that we discuss how we apply our mathematical knowledge and what assumptions we are making when we do so.

The slippery dip angle was 'safe'.

Surprisingly the answer was 30°. Ask students:

- *Why did we not spot that?* (We know the sin 30° but this is a different ratio.)

Calculate: $\cos 30 = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{2}{2.31} \approx 0.866$

Verify this using a calculator: find cos 30 ≈ ...

This is the cosine ratio: $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$

- *Is there another way to check?*

$\cos^{-1}\left(\frac{2}{2.31}\right)$

- *Would it have been possible to use the sine ratio to find the angle with this information?*

Students may realise they could use Pythagoras' Theorem to find the side opposite, but it would take longer and be less accurate. Encourage them to do so.

Making the most of mistakes: Create a language shift and a culture shift in your classroom

We can use activities such as this example to actively teach students that sometimes their deeper exploration of an idea will lead to an outcome that they hadn't expected. They have a choice to make when this happens. They can think:

- I was wrong. I'm not good at this.

OR

- I've changed my mind. I've learnt something new.

We can promote a growth mindset, by asking the class:

- Who changed their mind/discovered something new to them? Congratulations … you have just changed your brain!

Ask students to look at the third button on the calculator (tan), then ask:

- *Which sides might you divide for this ratio?*
- *How might we find out?*
  Students can experiment with the measurements from the 30° right-angled triangle and their calculator to see which combination verifies the tangent ratio.
  This is the tangent ratio: $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$
- *Why do you think there are only 3 ratios?* (There are, of course, 6 different combinations but 3 ratios cover every combination of 2 sides.)

Example 12: Set a problem for someone else

ACMMG224 ◆

Students apply trigonometry to solve right-angled triangle problems.

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/ relationships can you see?

Can you answer backwards questions?

Reasoning proficiency:

In what ways can you communicate?

What can you infer?

Instead of ***telling*** students about the cosine and tangent ratios, we can challenge students to identify them for themselves, by ***asking*** questions.

Challenge students to set (and solve) their own cosine and tangent ratio problem for other class members. This allows them to demonstrate that they understand how to use trigonometry to solve problems.

When students write their own problems for others to do, it demonstrates and consolidates their understanding. It is a type of backwards question and also serves to differentiate their learning. Students who understand the work well will try to set a complex question. Ask students:

- *What will you have to consider?*
- *Are there different types of problems? What information will you need to provide?*

This encourages students to identify the difference between when you are calculating a side and when you are calculating an angle. They might also realise that working out the length of the hypotenuse can be more difficult. Ask students:

- *Will the questions we make up always work? How will you know?*

Students who find this difficult might be encouraged to draw their triangle accurately and use these measurements. They still need to decide how much information is required and do the solutions to the task. There is another opportunity to differentiate learning when students swap questions.

Backwards questions

A ‘backwards’ question requires a little more thinking about than a standard application question. We ask this type of question to challenge students to work flexibly with a concept. There are many ways that teachers can do this. Two possible options include asking questions where students need to complete missing information and asking questions that start by giving the students an answer.

Example 13: Pythagoras’ short problems

ACMMG222 ◆ ◆
Students investigate Pythagoras’ Theorem and its application to solve simple problems involving right-angled triangles.

ACMMG224 ◆
Students apply trigonometry to solve right-angled triangle problems.

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/relationships can you see?

Can you represent/calculate in different ways?

Reasoning proficiency:

In what ways can you communicate?

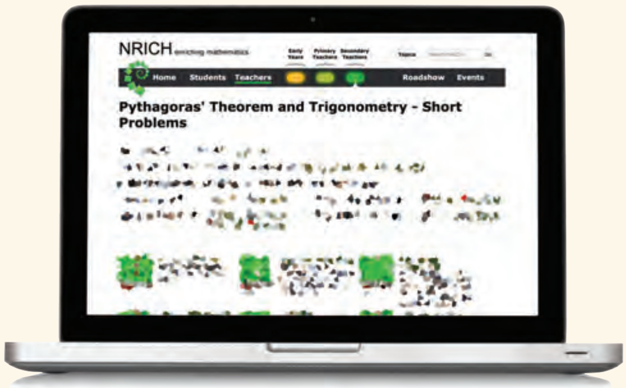
Instead of ***telling*** students to apply Pythagoras’ Theorem to solve the problems we can challenge them to identify them for themselves, by ***asking*** questions.

This activity is from the NRICH website.

This collection of problems offer an alternative to the standard application problems found in textbooks. The problems are described to the extent that it is not obvious that Pythagoras’ Theorem is to be used and the contexts are varied. When we ask students to solve these types of problems we are giving them the opportunity to work mathematically rather than ‘do maths’; asking them to show us what they can do rather than what they know.

When students are solving these problems, encourage them to ask questions such as:

- *What is the connection between…?*
- *Can you represent that algebraically?*
- *What else could it be?*
- *What are you being asked to find out?*



The link to these problems on the NRICH site is: http://nrich.maths.org/9344

Proficiency: Problem Solving

Proficiency emphasis and what questions to ask to activate it in your students (Examples 14–20)

Problems are described in the AC: Mathematics as 'meaningful' or 'unfamiliar'. Students of *all* abilities and ages should be provided with experiences of both meaningful and unfamiliar problems.

Meaningful problems

Meaningful problems are set in a context that a student can project themselves into. It may be that the mathematics and strategy being applied is familiar to the student or the problem relates to their own life experience. Connecting with a context does not mean that the students have to see it as 'fun' nor does it have to relate to an immediately practical situation from daily life.

Students can connect with a problem through provocations such as the use of manipulatives (either physical or digital) or through a dramatisation (eg, a story, interesting background information, a video clip). The intention is to give students the opportunity to work as a mathematician would work, in a context that they can access at their current stage of development.

Unfamiliar problems

Fundamentally there are two groups of unfamiliar problems:

- Problems for which the students would not be able to say that they had done a similar example previously, they would therefore need to create an approach (develop a strategy).
- Problems in which the students develop a new piece of knowledge. They begin the problem by applying the knowledge/skills that they have and they complete the problem having recombined that knowledge to form a new piece of understanding.

Growth mindset: Learning that not knowing is the beginning of a learning opportunity

Unfamiliar problems tend to provoke a response of, 'I don't know', or 'I'm not sure'. Students respond differently to this feeling; some shut down, others begin to ask, 'But how could I work that out?'

In developing powerful learners we are aiming for all of our students to learn that 'not knowing' is the *beginning of a learning opportunity* and that the first move that they need to make on the journey to finding out more is to ask, 'What could I do to work this out?'

Engaging in problem solving supports the move *from tell to ask*

Instead of ***telling*** students:

- the problem to solve
- the information they'll need
- the steps they should take.

We can ***ask students to identify***:

- the problem to solve
- the information they'll need
- a possible process to use.

Proficiency: Problem Solving examples	
Example 14: The box challenge Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.	ACMMG222 ◆ ◆
Example 15: Circle, square, circle Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.	ACMMG222 ◆ ◆
Example 16: Who gets to the taco cart first? Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.	ACMMG222 ◆
Example 17: Lie-thagoras' Theorem Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.	ACMMG222 ◆
Example 18: Water-tight proofs Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.	ACMMG222 ◆
Example 19: Circle, circle, rhombus Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.	ACMMG222 ◆
Example 20: Where is $\sqrt{2}$ on the number line? Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.	ACMMG222 ◆

Example 14: The box challenge

ACMMG222 ◆ ◆

Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.

Questions from the BitL tool

Problem solving proficiency:

Interpret; Model and plan; Solve and check; Reflect.

Reasoning proficiency:

What can you infer?

Instead of ***telling*** students to apply Pythagoras' Theorem to solve the problems we can challenge them to identify them for themselves, by ***asking*** questions.

Present a cardboard box to your students and ask them to cut the longest piece of rod that will fit into this box. Inform the students that they can't measure the box, but they can know its length, width and height. They get to cut and try the rod just once, so students need be sure that they are convinced about their solution before they make the cut.

Teachers or students could extend the problem into generalisation by asking:

- *Can I create a rule that will work for any rectangular prism?*

Interpret

What have you been asked to calculate? What information is helpful/no use? What additional information would you like? (Establish that the student is aware where the rod will need to be placed. Remember that telling students or prompting heavily is STEALING the opportunity for them to realise for themselves later on in the task.)

Model and plan

Do you have an idea? What could you try? What mathematics could help you with this problem? Is there more than one way that you could solve the problem? Would it help if you drew a sketch? (Ask students to speak to someone who they think is being a good problem solver today and ask them to show them what they are trying.)

Solve and check

Does that length seem about right to you? How could you check? Do other people think that too?

Questions to be used only after students have grappled with the problem for a few minutes:

How could you extend this problem? Could you create a rule that would work for any rectangular prism? What if you needed to position the rod in a different prism, for example, in a triangular, pentagonal or hexagonal prism (where the cross-section is a regular polygon). Would you be able to calculate the length of the rod if you were only given the length of the edges of the polygon and the depth of the prism?

Reflect

What was your most efficient method? Did other people solve this problem in a different way? Is there something that you would do differently next time? Will your method work for any values?

Example 15: Circle, square, circle

ACMMG222 ◆ ◆

Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.

Questions from the BitL tool

Problem solving proficiency:

Interpret; Model and plan; Solve and check; Reflect.

Reasoning proficiency:

What can you infer?

Instead of ***telling*** students to apply Pythagoras' Theorem to solve the problems we can challenge them to identify them for themselves, by ***asking*** questions.

Present Figure 17 to your students. Ask students to calculate the area contained between the two circles shown in this diagram. The challenge is that they can only ask for one measurement.

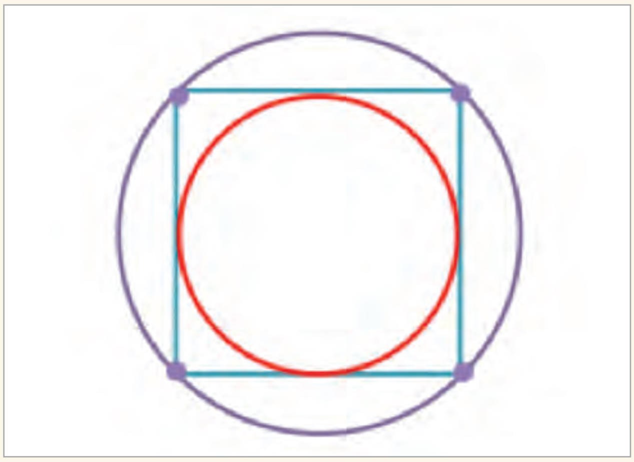


Figure 17

Ask students:

- *What measurement would you ask for? Why? Is there another solution?*

In this question, the students need to go through a process of 'if … then …' scenarios, such as:

- *If I knew the radius of the inner circle, then what would I be able to calculate?*
- *If I knew the diagonal of the square, what would I be able to calculate? Etc.*

We can support students by explaining the process of assuming that they have a piece of information and exploring the implications of that scenario.

This task can be extended to provide an opportunity for students to generalise. We could ask:

- *What relationships could you explore if you continued this pattern by drawing another square around the purple circle and then another circle around that square?*

Examples of relationships that could be explored would be:

- *What's the relationship between the radii of the circles?*
- *What's the relationship between the areas of the circles?*
- *What's the relationship between the areas of the squares?*

Further challenges lie in questions such as:

- *If the radius of the inner circle is 'r', what would the radius of the n^{th} circle be?*

Example 16: Who gets to the taco cart first?

ACMMG222 ◆ ◆

Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.

Questions from the BitL tool

Problem solving proficiency:

Interpret; Model and plan; Solve and check; Reflect.

Reasoning proficiency:

What can you infer?

Instead of ***telling*** students to apply Pythagoras' Theorem to solve the problems we can challenge them to identify them for themselves, by ***asking*** questions.

This activity is a Dan Meyer ***Three-Act Maths Task***. It can be presented to students along with the question, *What's the first question that comes to mind?*

This activity poses a problem about the shortest versus quickest path to the taco cart. The videos engage the students with a contextual problem. Not all the information is given to them at the start. They are required to consider rates as well as distances.

The **taco cart** activity can be accessed at: http://threeacts.mrmeyer.com/tacocart

Identifying the question to solve

The group can share questions and sort them into mathematical and non-mathematical questions. Then, of the mathematical questions, students can sort their questions into those that cannot be answered with the given information and those that could be answered using the given information or additional information that could be inferred.

Dan Meyer has a technique that we have seen many teachers adopt when generating and collecting questions from students. First, he asks students to individually write down questions that come to mind. Then, as he invites students to share their questions, he writes students' names next to the questions. He also asks if anyone else likes that question. *'Did you write it down, or if you didn't perhaps you still think that it's a good question.'* Through doing this, both Dan and his class get a sense of the questions that are of interest to the students.

Example 17: Lie-thagoras' Theorem

ACMMG222 ◆ ◆

Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.

Questions from the BitL tool

Problem solving proficiency:

Interpret; Model and plan; Solve and check; Reflect.

Reasoning proficiency:

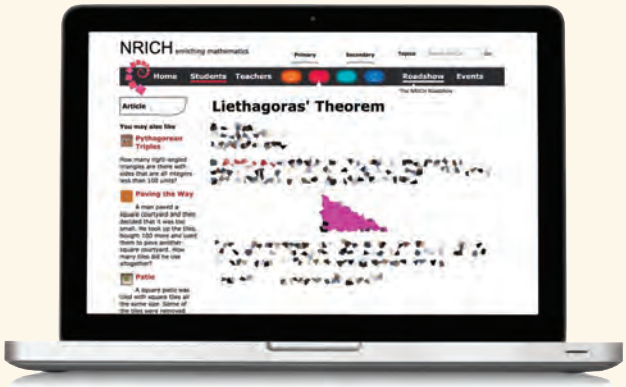
What can you infer?

Instead of ***telling*** students to apply Pythagoras' Theorem to solve the problems we can challenge them to identify them for themselves, by ***asking*** questions.

This activity is from the NRICH website.

In this activity, the students are asked to disprove a conjecture made by the fictitious cousin of Pythagoras.

The link to this problem on the NRICH site is: http://nrich.maths.org/2722



Conjecture: Verification, proof and disproof

It is important for students to realise that not every pattern that we identify that works for a few cases, will apply in all cases.

Some important ideas to develop about the way mathematicians work are:

- Mathematicians make conjectures or generalised statements from patterns they observe that may be true or not.
- No matter how many times they show it works for individual cases (ie they verify it), this does not **prove** that it works for all cases.
- To prove a conjecture to be true, they need to give a generalised argument that explains why it will always work that convinces others. (See Example 18 on page 29, James Garfield's proof.)
- You only need to find one case when it does not work to **disprove** the conjecture.
- If the conjecture has been shown not to work in all cases, mathematicians don't give up but try to find if there are special cases when it will work and then prove this to be true.

Example 18: Water-tight proofs

ACMMG222 ◆ ◆

Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.

Questions from the BitL tool

Problem solving proficiency:

Interpret; Model and plan; Solve and check; Reflect.

Reasoning proficiency:

What can you infer?

Instead of ***telling*** students to apply Pythagoras' Theorem to solve the problems we can challenge them to identify them for themselves, by ***asking*** questions.

Teachers can look for opportunities to ask students to prove their thinking to others or to themselves. Pythagoras' Theorem offers a wealth of opportunities for more formal algebraic proof. Googling 'proofs of Pythagoras' Theorem' will reveal many options for students to consider and many teachers will have their favourite proof.

The key is to find a way to present the opportunity for students to generate the proof (or proofs) for themselves. This is now a problem solving opportunity.

A different approach that activates students' reasoning, is to show an animation of a proof and ask students to explain/narrate the animation.

A demonstration using water that with no narration leads to some good discussion can be found at https://www.youtube.com/watch?v=CakMUdeB06o. Play the video then ask students:

- *Is this about area? How does it demonstrate Pythagoras' Theorem? Is it a proof?*
- *What do you notice about the angle the girl holds the apparatus at? Why?*

Examples of animations of two different (simple) proofs can be found at www.mathsisfun.com/pythagoras.html. Show the animations and ask students to:

- compare their proofs
- consider which is most convincing or easiest to understand.

One example of a proof is 'James Garfield's proof'.

Present Figure 18 to students, then ask:

- *What do you notice about the two triangles?*
- *How could you label the triangles to show your observations (about side length)?*
- *What's the size of the angle between the two triangles? What does it look like? Can you measure it? Will it still be 90 degrees if you enlarge the triangles? Is it possible to prove that the angle is 90 degrees?*

Students can be supported through a problem solving process to manage this part of the proof by the teacher asking questions such as:

Interpret

What are you being asked to show? What information could be useful? (Remember that telling students or prompting heavily is STEALING the opportunity for them to realise for themselves later on in the task.)

Model and plan

Would labeling your diagram help? Would it help to work with values before you try to work algebraically?

Solve and check

What do you know about angles in a triangle? What do you know about angles on a straight line?

Reflect

Did you expect this answer? Did other people use the same approach? How elegant was your strategy? Would you approach this differently next time?

Figure 18

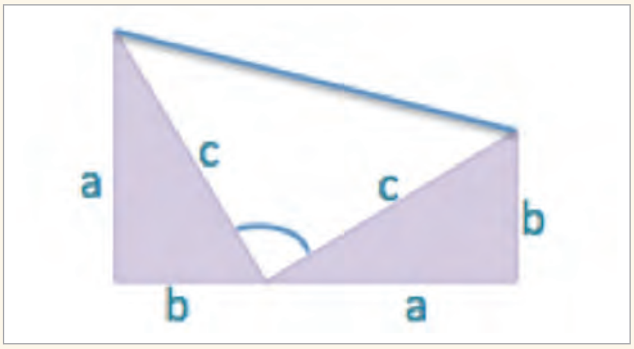


Figure 19

James Garfield (the 20th president of the United States) used the diagram in Figure 19 (on page 29) to create a proof of Pythagoras' Theorem.

Some questions that could help students if they are finding it difficult to begin their thinking are:

- *Garfield used area calculations to create his proof. Does this give you a new idea? What could you try now?*
- *Try calculating the area of the whole shape in two different ways. Does this give you new information to work with?*
- *If you are unsure how to calculate the area of a trapezium, could you put two trapezia together to form a different shape that you are more confident to work with, like a square? Does this give you a new idea? What could you try now?*
- *Think of '=' as meaning 'is the same as'. You have two different area calculations that are the same as each other. Does this give you a new idea? What could you try now?*

A collection of ways to engage students in proof of Pythagoras' Theorem can also be found on the NRICH site at http://nrich.maths.org/6553.

Example 19: Circle, circle, rhombus

ACMMG222 ◆ ◆

Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.

Questions from the BitL tool

Problem solving proficiency:

Interpret; Model and plan; Solve and check; Reflect.

Reasoning proficiency:

What can you infer?

Instead of ***telling*** students to apply Pythagoras' Theorem to solve the problems we can challenge them to identify them for themselves, by ***asking*** questions.

Present Figure 20 to your students. Two circles of the same diameter are made to overlap, and a rhombus is drawn, as shown in the diagram. Ask students, 'If you knew the area of the circle, under what circumstances would you be able to calculate the area of the rhombus?' Show how you would complete this calculation.

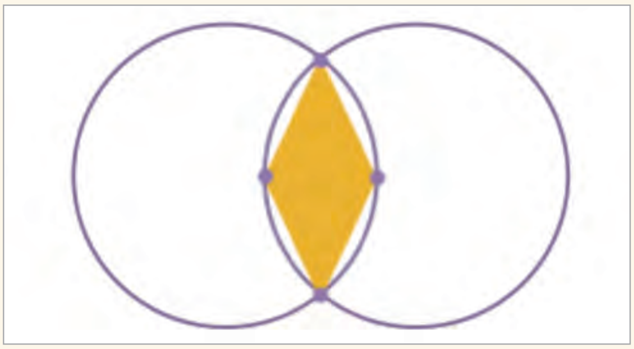


Figure 20

Example 20: Where is √2 on the number line?

ACMMG222 ◆ ◆

Students investigate Pythagoras' Theorem and its application to solve simple problems involving right-angled triangles.

Questions from the BitL tool

Problem solving proficiency:

Interpret; Model and plan; Solve and check; Reflect.

Reasoning proficiency:

What can you infer?

Instead of ***telling*** students to apply Pythagoras' Theorem to solve the problems we can challenge them to identify them for themselves, by ***asking*** questions.

In this activity, instead of telling students to apply Pythagoras' Theorem, we can challenge them to identify this for themselves. Ask students:

- *What two integers does √2 lie between on a number line? How do you know?*

Show students an example of a work sample, like the one in Figure 21, then ask:

- *How might you check whether the location is correct using this method?*
- *Can you explain the student's thinking?*
- *Now that you have seen this, what other irrational numbers might you locate on the number line using this method?*

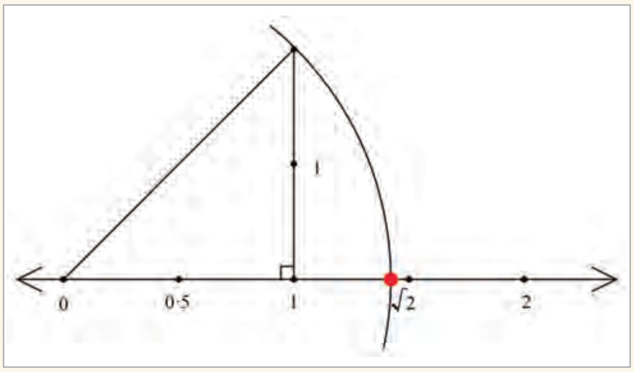


Figure 21

Connections between ‘Pythagoras and trigonometry’ and other maths content

There are many opportunities to connect to other content in the AC: Mathematics, when we use Pythagoras and trigonometry as a starting point.

Here are just some of the possible connections that can be made:

Mathematics: Year 8–9	
Whilst working with Pythagoras and trigonometry, connections can be made to:	**How the connection might be made:**
Investigate the concept of irrational numbers, including π. (Year 8) ACMNA186	Expressing lengths of sides exactly as surds, refer to: **Example 6: Why can’t I…?** **Example 7: Constructing a volleyball court** **Example 20: Where is √2 on the number line?**
Solve a range of problems involving rates and ratios, with and without digital technologies. (Year 8) ACMNA188	Solving trigonometric equations using the ratio of sides, refer to: **Example 5: A piece of knotted rope** **Example 8: A special ratio** **Example 9: Connecting student understanding to mathematical protocols** **Example 10: What about the triangles we rejected?** **Example 11: What if we don’t know the side opposite?**
Use the enlargement transformation to explain similarity and develop the conditions for triangles to be similar. ACMMG220	Similar right-angle triangles in understanding trigonometry, refer to: **Example 8: A special ratio**
Solve problems using ratio and scale factors in similar figures. ACMMG221	Identifying the trig ratio for similar right-angle triangles, refer to: **Example 8: A special ratio**
Solve problems involving direct proportion. Explore the relationship between graphs and equations corresponding to simple rate problems. ACMNA208	Using trigonometric ratios to find sides, refer to: **Example 8: A special ratio** **Example 9: Connecting student understanding to mathematical protocols** **Example 10: What about the triangles we rejected?** **Example 11: What if we don’t know the side opposite?**
Calculate areas of composite shapes. ACMMG216	Refer to: **Example 15: Circle, square, circle** **Example 19: Circle, circle, rhombus**

Making connections to other learning areas

We know that when our students meet a concept frequently and in different contexts, they have a greater chance of developing understanding. With this in mind, it is our responsibility to help our students to make these connections by intentionally designing tasks that connect a number of different content descriptions. Alternatively, connections can be made through questioning individual or small groups of students.

‘Pythagoras and trigonometry’ from Year 9 to Year 10A

The AC: Mathematics year level content descriptions shown here have been colour coded to highlight the following curriculum aspects of working with Pythagoras and trigonometry:

Investigating and understanding similarity, Pythagoras’ Theorem and/or trigonometry ◆
In **Year 9 to Year 10** students investigate for conceptual understanding and establish the trigonometric rules for finding areas of any triangles

Using and applying similarity, Pythagoras’ Theorem and/or trigonometry to solve problems ◆
In **Year 10 to Year 10A** students apply their knowledge of Pythagoras and trigonometry to solve increasingly complex problems in increasingly abstract situations.

Year level	‘Pythagoras and trigonometry’ content descriptions from the AC: Mathematics
Year 9 ◆	Students investigate Pythagoras’ Theorem and its application to solve simple problems involving right-angled triangles. ACMMG222
Year 9 ◆	Students use similarity to investigate the constancy of the sine, cosine and tangent ratios for a given angle in right-angled triangles. ACMMG223
Year 9 ◆	Students apply trigonometry to solve right-angled triangle problems. ACMMG224
Year 10 ◆	Students solve right-angled triangle problems including those involving direction and angles of elevation and depression. ACMMG245
Year 10A ◆ ◆	Students establish the sine, cosine and area rules for any triangle and solve related problems. ACMMG273
Year 10A ◆ ◆	Students use the unit circle to define trigonometric functions, and graph them with and without the use of digital technologies. ACMMG274
Year 10A ◆	Students solve simple trigonometric equations. ACMMG275
Year 10A ◆	Students apply Pythagoras’ Theorem and trigonometry to solve three-dimensional problems in right-angled triangles. ACMMG276

Numeracy continuum: Using spatial reasoning	
End Foundation	Visualise 2D shapes and 3D objects: sort and name simple 2D shapes and 3D objects.
End Year 2	Visualise 2D shapes and 3D objects: identify, sort and describe common 2D shapes and 3D objects.
End Year 4	Visualise 2D shapes and 3D objects: visualise, sort, identify and describe symmetry, shapes and angles in the environment.
End Year 6	Visualise 2D shapes and 3D objects: visualise, sort, describe and compare the features of objects such as prisms and pyramids in the environment.
End Year 8	Visualise 2D shapes and 3D objects: visualise, describe and apply their understanding of the features and properties of 2D shapes and 3D objects.
End Year 10	Visualise 2D shapes and 3D objects: visualise, describe and analyse the way shapes and objects are combined and positioned in the environment for different purposes.

Source: ACARA, Australian Curriculum: Mathematics, Version 8.1

Resources

NRICH website

http://nrich.maths.org

In this conceptual narrative we have highlighted the possibility of using tasks from an organisation called NRICH enriching mathematics.

The NRICH website contains a large collection of high quality maths problem solving tasks, together with suggestions about content that may be related to the task, ways to get started and different (valid) solutions that have been submitted by students from around the world.

Dan Meyer's blog: 101 questions

http://www.101qs.com

Dan's blog contains images and short films that can be presented to students along with the question: *What's the first question that comes to mind?*

A spreadsheet of ***Dan Meyer's Three-Act Maths Tasks*** can be accessed at http://bit.ly/DM3ActMathTasks.

Notes

Do you want to feel more confident about the maths you are teaching?

Do you want activities that support you to embed the proficiencies?

Do you want your students thinking mathematically rather than just doing maths?

If you answered **yes** to any of these questions, then this resource is for you.

Packed full of examples, along with questions you can ask students as they engage in their learning, this resource supports you to develop confidence in teaching the Australian Curriculum: Mathematics.

Produced by Office for Education and Early Childhood