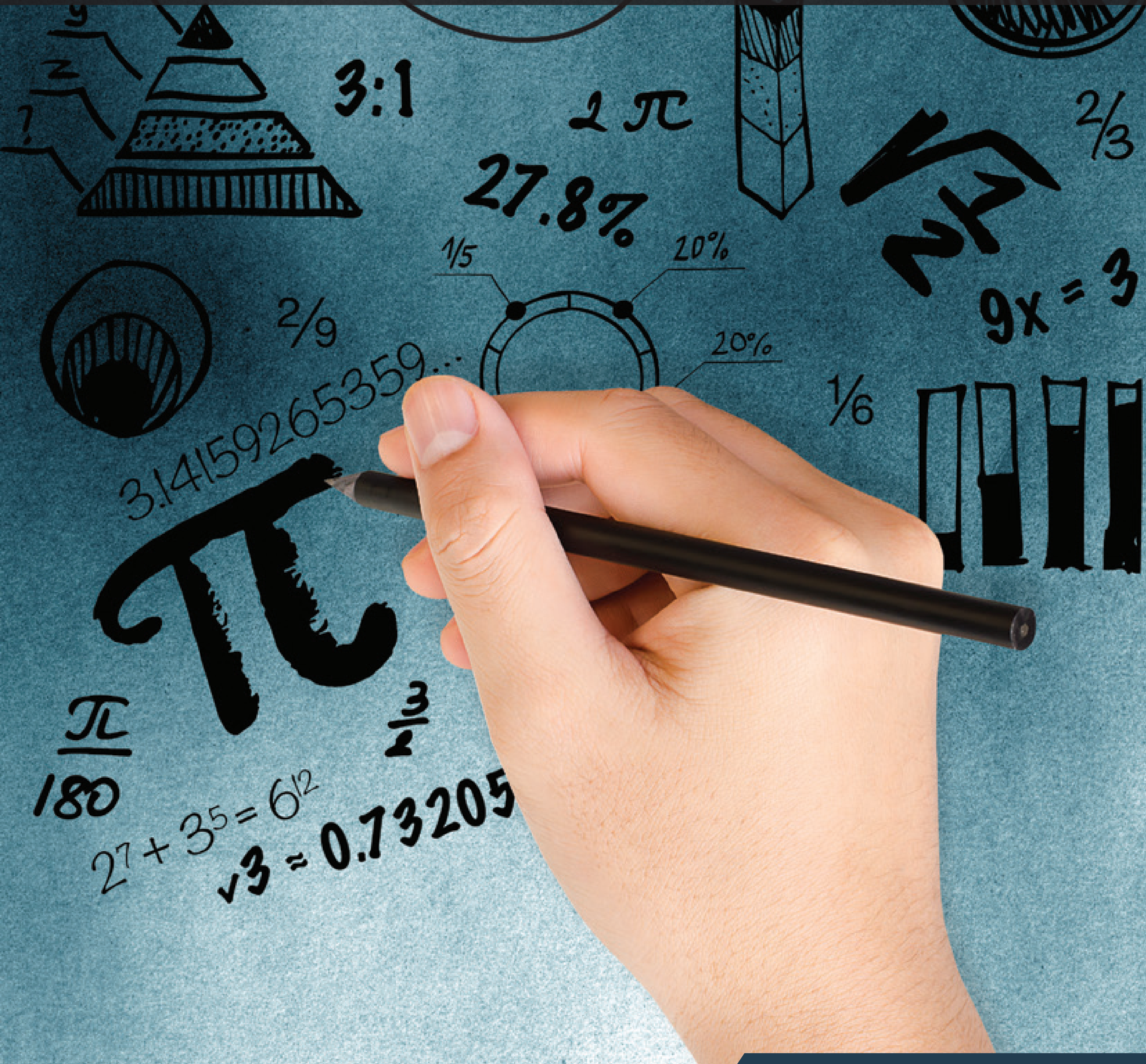
Real numbers: Year 8

MATHEMATICS CONCEPTUAL NARRATIVE

Leading Learning: Making the Australian Curriculum work for us
by bringing CONTENT and PROFICIENCIES together



www.acleadersresource.sa.edu.au

Government of South Australia
Department for Education

Contents

Resource key

The '**AC**' icon indicates the Australian Curriculum: Mathematics content description(s) addressed in that example.

The '**From tell to ask**' icon indicates a statement that explains the transformation that is intended by using the task in that example.

More information about 'Transforming Tasks':
http://www.acleadersresource.sa.edu.au/index.php?page=into_the_classroom

Look out for the purple pedagogy boxes, that link back to the SA TfEL Framework.

The '**Bringing it to Life (BitL)**' tool icon indicates the use of questions from the *Leading Learning: Making the Australian Curriculum Work for Us* resource.

Bringing it to Life (BitL) key questions are in bold orange text.

Sub-questions from the BitL tool are in green medium italics – these questions are for teachers to use directly with students.

More information about the 'Bringing it to Life' tool:
http://www.acleadersresource.sa.edu.au/index.php?page=bringing_it_to_life

Throughout this narrative—and summarised in **'Real numbers' from Year 7 to Year 10A** (see page 25)—we have colour coded the AC: Mathematics year level content descriptions to highlight the following curriculum aspects of working with real numbers:

- Recognise, describe and represent real numbers
- Compare and order real numbers
- Convert and calculate using real numbers
- Apply and solve problems using real numbers.

What the Australian Curriculum says about ‘Real numbers’

Content descriptions

Strand | Number and algebra.

Sub-strand | Real numbers.

Year 8 ◆ | ACMNA184
Students investigate terminating and recurring decimals.

Year 8 ◆ | ACMNA186
Students investigate the concept of irrational numbers, including π.

Year 8 ◆ | ACMNA187
Students solve problems involving the use of percentages, including percentage increases and decreases, with and without digital technologies.

Year 8 ◆ | ACMNA188
Students solve a range of problems involving rates and ratios, with and without digital technologies.

Year level descriptions

Year 8 ◆ | Students describe patterns involving indices and recurring decimals.

Year 8 ◆ ◆ | Students calculate accurately with simple decimals, indices and integers, recognise equivalence of common decimals and fractions including recurring decimals.

Year 8 ◆ | Students formulate, and model practical situations involving ratios.

Achievement standards

Year 8 ◆ | Students solve everyday problems involving rates, ratios and percentages.

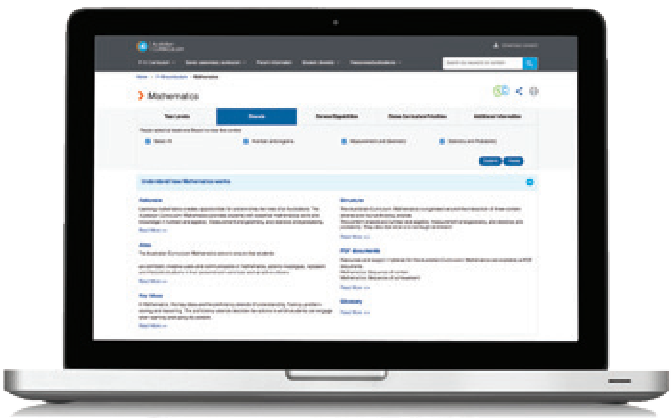
Year 8 ◆ | Students describe rational and irrational numbers.

Year 8 ◆ | Students solve problems involving profit and loss.

Numeracy continuum

Using fractions, decimals, percentages, ratios and rates

End of Year 8 ◆ | Students visualise and describe the proportions of percentages, ratios and rates (Interpret proportional reasoning). Students solve problems using simple percentages, ratios and rates (Applying proportional reasoning).



Source: ACARA, Australian Curriculum: Mathematics

Working with Real numbers

Important things to notice about this sub-strand of the Australian Curriculum: Mathematics and numeracy continuum

What we are building on and leading towards in Year 8 'Real numbers'

In Year 7 students convert between any fraction, decimal and percentage. Students develop strategies to multiply and divide fractions and decimals, rounding to a given number of decimal places.

In Year 8 students investigate terminating and recurring decimals, introducing the concept of irrational numbers and linking this to the development of an understanding of π.

In Year 9 students build on their fluency in multiplying and dividing decimals by powers of ten and begin to express numbers in scientific notation. An application of this is seen in 'Using units of measurement' when Year 9 students investigate large and small time scales.

In Year 10A students would be expected to define rational and irrational numbers.

- While there are some standard mathematical notations used worldwide there are still some cultural differences with the ways of describing a recurring decimal. 0.235235235 … could be written as $0.\dot{2}3\dot{5}$ or 0.[235] but is most commonly written as $0.\overline{235}$ in South Australian schools.
- A common confusion for students is the difference between **part:part** and **part:whole** ratios as a comparison. Describing mixtures, such as those in 'Example 8: Strong cordial – ratios' on page 18, in both ways helps clarify this concept. For example, in a cordial mixture which is one third syrup, the ratio of syrup to water is 1:2.

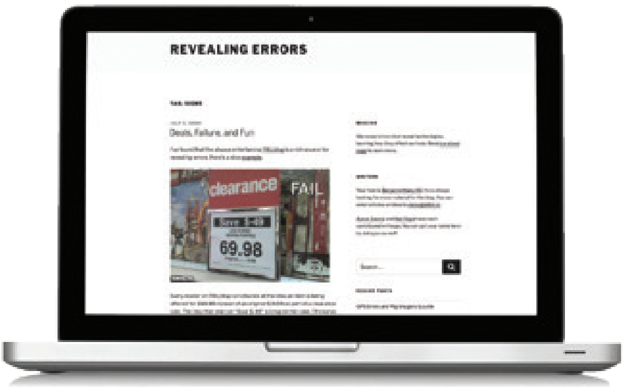
Engaging learners

Classroom techniques for teaching Real numbers

Revealing errors

Challenge students to identify numerical errors in signs, advertisements or the media. The blog, **Revealing errors** has numerous examples.

The **Revealing errors** blog can be found at: https://revealingerrors.com/tags/signs



Squirt

Digital learning activities can be used in conjunction with concrete materials to hook students into their learning. A useful example is the digital activity **Squirt**. Being able to mix, pour and observe gives students a practical experience that supports conceptual understanding of ratios and rates.

The **Squirt** activity can be found at: http://www.scootle.edu.au/ec/viewing/L1996/index.html

Source: *Squirt*, © Education Services Australia Ltd, 2013

Estimation 180

Estimating with ratios encourages students to connect their intuitive knowledge about ratios to practical situations, building fluency and conceptual understanding.

Estimation 180 is a website that has multiple scenarios that can prompt students to estimate, inviting every learner to have an opinion. This resource could be used as a warm-up to lessons, choosing contexts that build their number sense.

The **Estimation 180** activites can be found at: http://www.estimation180.com/day-4.html

Source: *Estimation 180: Building number sense one day at a time,* by Andrew Stadel

Rope around the world conundrum

The **Rope around the world** activity is a fascinating, counter-intuitive puzzle:

Consider a belt tied around the Equator of the Earth. If I increased the diameter of the Earth by one metre, how much longer would the belt need to be?

Consider a piece of string tied around the middle of a soccer ball. If I increased the diameter of the soccer ball, how much longer would the piece of string need to be?

Do you believe it is the same in both cases and it is just over 3 metres? Why is it so?

The **Rope around the world** activity can be found at: http://www.abc.net.au/science/surfingscientist/pdf/conundrum17.pdf

Source: *Conundrum 17: Rope around the world*, The Surfing Scientist, © 2005 Ruben Meerman, ABC Science Online

From tell to ask

Transforming tasks by modelling the construction of knowledge (Examples 1–7)

The idea that education must be about more than transmission of information that is appropriately recalled and applied, is no longer a matter for discussion. We know that in order to engage our students and to support them to develop the skills required for success in their life and work, we can no longer rely on a 'stand and deliver' model of education. It has long been accepted that education through transmission of information has not worked for many of our students. Having said this, our classrooms do not necessarily need to change beyond recognition. One simple, but highly effective strategy for innovation in our classrooms involves asking ourselves the question:

What information do I need to tell my students and what could I challenge and support them to develop an understanding of for themselves?

For example, no amount of reasoning will lead my students to **create irrational numbers** for themselves. They need to receive this information in some way. However, it *is* possible for my students to be challenged with a question that will result in them identifying the **concept of an irrational number**, so I don't need to instruct that information.

At this stage of development, students can **develop an understanding of real numbers when generating and matching fractions, decimals and percentages**. When teachers provide opportunities for students to **identify and describe the properties of numbers, with and without technology**, they require their students to generalise from the classifications they have made. Telling students the properties removes this natural opportunity for students to make conjectures and verify connections that they notice.

When we are feeling 'time poor' it's tempting to believe that it will be quicker to tell our students a formula, rather than ask a question (or series of questions) and support them to establish a formula for themselves. Whether this is true or not really depends on what we have established as our goal. If our goal is to have students recall and apply a particular rule or theorem during the current unit of work, then it probably is quicker to tell them the rule and demonstrate how to apply it. However, when our goal extends to wanting students to develop conceptual understanding, to learn to think mathematically, to have a self-concept as a confident and competent *creator* and *user* of mathematics, then telling students the rule is a false economy of time.

When we challenge our students to **establish a theorem**, we model that algebra can be powerful and useful. We provide our students with an authentic context for working algebraically. Telling students formulae removes this opportunity for students to generalise.

Curriculum and pedagogy links

The following icons are used in each example:

The '**AC**' icon indicates the Australian Curriculum: Mathematics content description(s) addressed in that example.

The '**Bringing it to Life (BitL)**' tool icon indicates the use of questions from the *Leading Learning: Making the Australian Curriculum Work for Us* resource.

The Bringing it to Life tool is a questioning tool that supports teachers to enact the AC: Mathematics Proficiencies: http://www.acleadersresource.sa.edu.au/index.php?page=bringing_it_to_life

The '**From tell to ask**' icon indicates a statement that explains the transformation that is intended by using the task in that example.

This idea of moving 'From tell to ask' is further elaborated (for Mathematics and other Australian Curriculum learning areas) in the 'Transforming Tasks' module on the *Leading Learning: Making the Australian Curriculum work for Us* resource: http://www.acleadersresource.sa.edu.au/index.php?page=into_the_classroom

Look out for the purple pedagogy boxes, that link back to the SA TfEL Framework.

From tell to ask examples	
Example 1: Real number card sort Students investigate terminating and recurring decimals. Students investigate the concept of irrational numbers, including π.	ACMNA184 ◆ ACMNA186 ◆
Example 2: Terminating or recurring Students investigate terminating and recurring decimals.	ACMNA184 ◆
Example 3: What number times by itself gives …? Students investigate the concept of irrational numbers, including π.	ACMNA186 ◆
Example 4: Carpet squares – square roots Students investigate the concept of irrational numbers, including π.	ACMNA186 ◆
Example 5: Paper sizes – irrational numbers Students investigate the concept of irrational numbers, including π.	ACMNA186 ◆
Example 6: Hardware ratios Students solve a range of problems involving rates and ratios, with and without digital technologies.	ACMNA188 ◆
Example 7: Calculating Pi Students investigate the concept of irrational numbers, including π. Students solve problems involving the use of percentages, including percentage increases and decreases, with and without digital technologies.	ACMNA186 ◆ ACMNA187 ◆

Example 1: Real number card sort

ACMNA184 ◆
Students investigate terminating and recurring decimals.

ACMNA186 ◆
Students investigate the concept of irrational numbers, including π.

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/relationships can you see?

Can you represent/calculate in different ways?

Reasoning proficiency:

In what ways can your thinking be generalised?

In what ways can you communicate?

Instead of ***telling*** students about different types of real numbers, we can challenge students to recognise the properties for themselves, by ***asking*** questions.

For this activity, use a set of real number cards with both rational (numbers that can be written as a fraction using whole numbers, for example. $^-4$, $1\frac{2}{7}$, $0.\overline{4}$) and irrational numbers (surds and non-recurring decimals, for example $\sqrt{3} \approx 0.73205\ldots$, π, e). Ask students to sort them in some way and describe the properties or characteristics of each group. Include multiple representations of numbers such as the different ways to represent a recurring decimal. Even those students who do not know about irrational numbers are able to identify similarities about the numbers and this will encourage them to find out more about them. Ask students:

- *Why did you group the cards in that way? Can you describe the characteristics of these numbers in as many ways as possible?*
- *Did any other groups sort the cards in the same way? Can you make up a card for another number that might belong in this group?*
- *Does anyone know the mathematical name of that group of numbers? How could we find out? How could we check?*

  To promote a community of learning, you might suggest students ask senior students or other mathematics teachers to learn more about irrational numbers. They could also use a mathematics dictionary.

As the different types of numbers are identified, construct a Venn diagram on a display board (see Figure 1) as students identify the relationships of the number types.

There are many different ways to represent the number system.

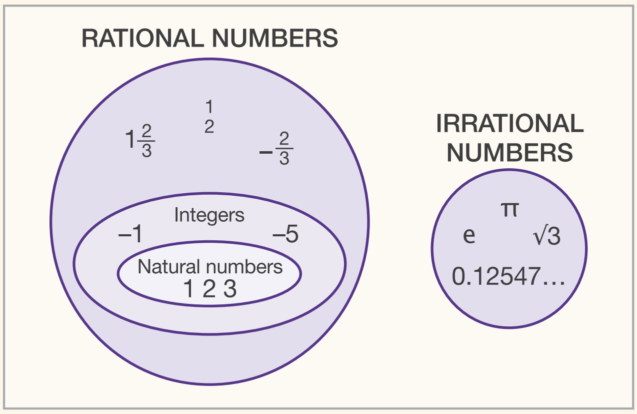


Figure 1

It is important that the diagram the students draw represents their own sorting and that it is a consistent representation. For example, there are no numbers which are both rational and irrational and so the sets are disjointed sets. If students do draw them as intersecting, ask if it is possible to have a number in that area, and if not, how might they redraw the diagram to show this.

Going from the known to the unknown:
Getting students to demonstrate what they already know and understand

When students have sorted the cards, they are being asked to notice. Asking students to sort and describe the numbers before they name them, focuses on the concept before the terminology and formal definition. All students have an entry point to the learning with natural numbers, integers, fractions and decimals. This process also identifies what type of numbers we want to know more about.

Example 2: Terminating or recurring

ACMNA184 ◆

Students investigate terminating and recurring decimals.

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/ relationships can you see?

Can you represent/calculate in different ways?

Reasoning proficiency:

In what ways can your thinking be generalised?

Instead of ***telling*** students about different types of real numbers, we can challenge students to recognise the properties for themselves, by ***asking*** questions.

In groups, ask students to change unit fractions; for example $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}$ into decimals. Ask students:

- *What different methods did you use?* (Students may use different methods for different fractions. $\frac{1}{2}, \frac{1}{4}, \frac{1}{10}$ are likely to be known facts, and you may choose to get them to explain why they have those decimal values. Students may change the fractions to denominators of 10, 100, … powers of 10, divide the numerator by the denominator, or use a calculator.)
- *What do you notice? In what ways are the decimals similar, different?* (For the recurring decimals such as $\frac{1}{7}$, it will be important to check that students realise that if they write the decimal to a finite number of decimal places, that it is not exactly $\frac{1}{7}$. Using a calculator 1÷7 shows the 0.1428571 … depending on the screen spaces. It does, however, store the exact value of $\frac{1}{7}$ and if you multiply by 7, the result will be 1. If however, you clear the screen and type in 0.1428571 and multiply it by 7, the result is 0.9999997 which is close to 1 but not exact because the decimal you typed in is not exactly $\frac{1}{7}$.)
- *If you were told that these decimals can be sorted into two groups, terminating decimals and recurring decimals, how might you do that? Check the sort with another group.*
- *What do you understand by terminating decimals and recurring decimals? How would you check this understanding?*

For the following unit fractions $\frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \frac{1}{15}, \frac{1}{16}, \frac{1}{17}, \frac{1}{18}, \frac{1}{19}, \frac{1}{20}$, ask students:

- *Which ones might be terminating, which recurring? Why do you think that? How could you check this?* ($\frac{1}{12}, \frac{1}{16}, \frac{1}{20}$. Numbers which are factors of, and hence can be changed to: 10, 100, 1000 or other powers of 10 by multiplication by a whole number, will be terminating decimals because these will have place values in the decimal system.)
- *Now that you know this, can you predict in general, which other types of fractions might be terminating?*
- *If you know that $\frac{1}{50}$ is a terminating decimal, what do you think $\frac{2}{50}, \frac{3}{50}, \frac{4}{50}$, etc might be? Why?*
- *What other questions come to mind?*

Example 3: What number times by itself gives …?

ACMNA186 ◆
Students investigate the concept of irrational numbers, including π.

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/ relationships can you see?

Can you represent/calculate in different ways?

Reasoning proficiency:

In what ways can your thinking be generalised?

Instead of ***telling*** students about square roots which are irrational, we can challenge students to recognise the properties for themselves, by ***asking*** questions.

For this activity, ask students to make up some questions for Year 7 students that begin with 'What number times by itself gives …?' Ask them to make up at least two of each type that Year 7 students might find 'very easy', 'more challenging' and 'very difficult' and be prepared to justify their choice. Students can write their questions on cards and put what they think might be the answer on the back and place them in three lucky dip boxes labelled in the categories above.

As a class discussion, draw cards from the boxes starting with the 'very easy', and ask students to write their answers on their mini-whiteboard or equivalent, to determine students' current understanding. Identify whose question it was and why they considered it very easy. This process is likely to identify perfect squares, which can be displayed in the room. The 'more challenging' questions might include larger numbers, fractions or decimals.

In the 'very difficult' box, the answers are likely to be irrational numbers. For these questions, ask students:

- *What might the answer be? What isn't it? Can you give a range of values?* (What number times by itself gives 7? As 2 x 2 = 4 and 3 x 3 = 9, we can deduce that the answer would be between 2 and 3.)

Identify if any of the students were able to suggest an answer, then ask:

- *Can you explain your thinking? How might we check this answer? Does that make sense?*

Consider another difficult question and encourage students to use a method they have heard of (eg using the square root function on the calculator, or another). Most students will use trial and error with a calculator. For students who understand the concept of a square root, differentiate the task by asking:

- *What number times by itself 3 times gives 7?*
- *How can you use our previous thinking to help with this problem?*

Once the students have an understanding of what a square root is, introduce the symbol and confirm that it is asking the same question, using the calculator to find the square root of perfect squares from the 'very easy' box. The calculator can also be used to verify the approximate values students found by trial and error. Explore other buttons on the calculator that might do the same thing (7^ ($\frac{1}{2}$)). Have a discussion about accuracy by asking:

- *How might you help this student with their thinking?*
- *When I type in $\sqrt{2}\times\sqrt{2}$ on the calculator I get exactly 2 but when I use the value I found by trial and error 1.414 x 1.414, I get 1.999 something. What am I doing wrong?*

Example 4: Carpet squares – square roots

ACMNA186 ◆
Students investigate the concept of irrational numbers, including π.

Questions from the BitL tool

Understanding proficiency:
What patterns/connections/ relationships can you see?
Can you represent/calculate in different ways?

Reasoning proficiency:
In what ways can your thinking be generalised?

Instead of ***telling*** students about square and cube roots which are irrational, we can challenge students to recognise the properties for themselves, by ***asking*** questions.

A rug manufacturer wants to make a range of square rugs. He has always made rugs with areas of 4, 9 and 16 square metres; but now wants to make rugs with a range of different areas. Ask students:

- *How might he represent these 3 rugs on his website?*
- *How might you work out the actual dimensions of these rugs?*
- *Why do you think he might have made square rugs with these areas and not rugs with areas of 2, 3 or 5 square metres?*

The manufacturer wants at least 10 different sizes with whole number areas, between 4 and 20 square metres. Ask students:

- *What dimensions would you suggest to satisfy the manufacturer requirements and appeal to customers? Explain your thinking.*

Example 5: Paper sizes – irrational numbers

ACMNA186 ◆
Students investigate the concept of irrational numbers, including π.

Questions from the BitL tool
Understanding proficiency:
What patterns/connections/relationships can you see?
Can you represent/calculate in different ways?
Reasoning proficiency:
In what ways can you communicate?
In what ways can your thinking be generalised?

Instead of ***telling*** students about irrational numbers, we can challenge students to recognise their significance through investigating similar shapes for themselves, by ***asking*** questions.

Begin a discussion about paper sizes that the students may be familiar with. Ask students:

- *What different paper sizes are you familiar with? What are they used for? Why?*

Issue the students with an A3, A4 and A5 sheet of paper, scissors, rulers and sticky tape. Ask students:

- *What is the same/different about these sheets of paper?* (Students often identify an **additive** relationship between the different sizes, for example, 'The difference goes up each time'. Encourage students to find a **multiplicative** relationship.)
- *If you know the measurements of these three sheets, what might be the size of A2 and A6 sheets? Why?*
- *What might be an estimate of the size (dimensions) and area of an A0 size sheet? How could you check?*
- *What is the scale factor for the enlargement for A5 to A4, A4 to A3? Can you generalise the relationship between 'consecutive' paper sizes, for example, An and A(n+1)?* (The area doubles but the scale factor is $\sqrt{2}$. This concept is encountered in both Example 1 and Example 4 in the *Geometric reasoning: Year 9* narrative and is a good connection with irrational numbers.)

For students who are seeking an intellectual challenge, ask:

- *Are paper sizes the same all over the world?*
- *What is the difference/same about paper sizes A, B and C? Are they similar rectangles? Can you generalise the ratios between the paper sizes B and C?*

Links to authentic contexts

Using examples/analogies from real life situations engages students and can contextualise the purpose of a mathematical process. Students are most likely to be familiar with the A5, A4 and A3 sizing of paper; but may not be aware of the fact that they are similar rectangles.

(This activity also appears in the *Geometric reasoning: Year 9* and *Real numbers: Year 10/10A* narratives.)

Example 6: Hardware ratios

ACMNA188 ◆
Students solve a range of problems involving rates and ratios, with and without digital technologies.

Questions from the BitL tool

Understanding proficiency:
What patterns/connections/ relationships can you see?
Can you represent/calculate in different ways?

Reasoning proficiency:
In what ways can your thinking be generalised?

Instead of ***telling*** students about the linear relationships in equal ratios, we can challenge students to explore the concepts for themselves, by ***asking*** questions.

This digital activity from the Scootle website provides practice and flexibility when working with ratios, but more importantly makes links with the graphical representation of equal ratios as a linear relationship.

Before beginning the task, ask students if they have ever fertilized their lawn at home or thought about how the groundsman might fertilise the school oval, or how the curator of Adelaide Oval might fertilise the turf. In some cases, it comes as a liquid concentrate. Ask students:

- *Why is it sold as a concentrate? What other things do you know are sold this way?*
- *What must be done before it can be used on the lawn?*
- *Why is it important to mix the fertiliser according to the instructions?*

It is important that students reflect on the experience of using this digital activity. Ask them:

- *How might you use this type of thinking in other situations?*

Ask students to write a memo for the new employee who is starting tomorrow, to help them calculate the quantities of concentrate in more than one way.

The **In proportion: graphs of ratios, rates and scales** activity requires a login to Scootle and can be accessed at: http://www.scootle.edu.au/ec/viewing/L8103/index.html

Links to authentic contexts

Using examples/analogies from real life situations engages students and can contextualise the purpose of a mathematical process. Encouraging students to share their prior knowledge and experiences can give those students who do not often contribute to class discussion in mathematics, to speak with confidence.

Example 7: Calculating Pi

ACMNA186 ◆
Students investigate the concept of irrational numbers, including π.

ACMNA187 ◆
Students solve problems involving the use of percentages, including percentage increases and decreases, with and without digital technologies.

Questions from the BitL tool

Understanding proficiency:
What patterns/ connections/ relationships can you see?

Instead of ***telling*** students the formula for the circumference of a circle, we can challenge students to explore the connection between circumference and diameter, and hence establish the formula by ***asking*** questions.

The main learning intention of this activity is to investigate the relationship between the circumference and diameter of a circle. It is included here as an example of how percentage increase and decrease can be authentic learning within other contexts, in particular when collecting continuous data.

Consider a race from one red post to another and then back, as represented in Figure 2. You can choose to run around the sides of a square (as shown in orange), take a circular path (blue), or take a straight path over and back (green). This can also be done by drawing these paths in chalk on the pavement. Ask students:

- *Which path would you take and why?* (This discussion might include practical factors, such as the time wasted on the direct path having to slow down, stop and accelerate; while the circular path can be run at a constant speed.)

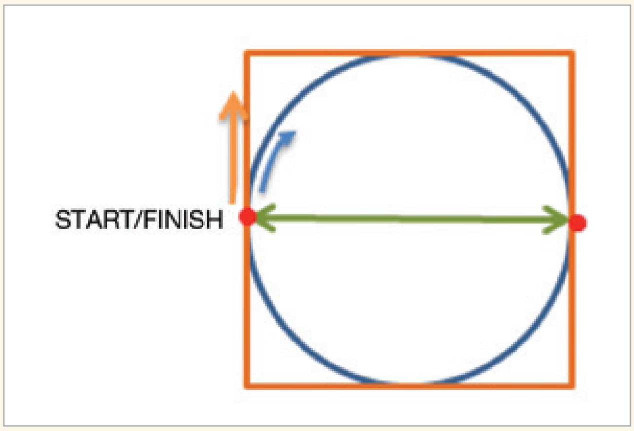


Figure 2

Consider the length of the different paths:

- *Which is the shortest/the longest path?*
- *If the distance between the two posts is 100 metres, can you determine the lengths of any of the paths accurately? For any of the paths you can't, can you make any statement about their length?* (Having a dimension for the circle provides a more concrete way of thinking about this problem. It also allows students to make an estimate for the circumference of the circle before they are asked to generalise for any diameter.)
- *If the distance between the two posts is d metres, can you determine an expression for the lengths of any of the paths accurately? For any of the paths you can't, can you make any statement about their length?* (We are looking for students to be able to make a statement such as, *'The square is 4 times the diameter (4d) the straight path is 2d and the circular path is less than 4d, but more than 2d'.*)

Some students may believe that the relationship will change depending on the size of the circle. Ask them:

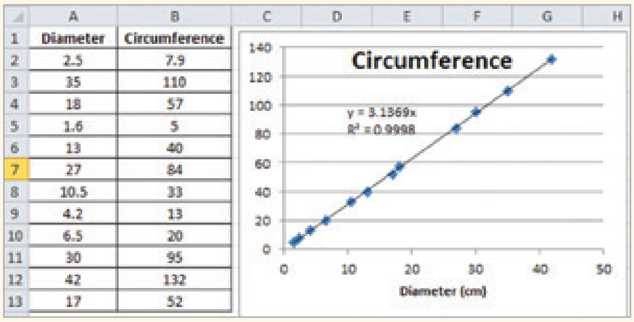
- *What if we use a bigger/smaller circle? What do you think the relationship will be then?* (The square will always be twice the distance of the direct path, but the difference in the distance run will be greater for larger diameters ($4d–2d=2d$ which gets bigger as d gets bigger).)

Students could begin to explore this problem using a range of diameters. It is easy to calculate the distance for the direct route and the square path; but how might you get the most accurate measures for the circular path, as all we know is that it will be between $2d$ and $4d$.

For students who already know about π and the formula for circumference, ask them why it works. How did people find this out? Ask them to convince you:

- *How might you change the method for measuring circumference and increase the accuracy of your measurements?* (Software could be used for a high degree of accuracy, for example Excel or CAD/Blender, which also links to Technology.)

The relationship between diameter and circumference could be plotted on a graph and the equation of the line could be investigated. This can be done manually or by using Excel and 'Fitting a trendline' (see Figure 3).



	A	B
1	Diameter	Circumference
2	2.5	7.9
3	35	110
4	18	57
5	1.6	5
6	13	40
7	27	84
8	10.5	33
9	4.2	13
10	6.5	20
11	30	95
12	42	132
13	17	52

Figure 3

A similar investigative approach could be taken to establish the relationship between radius and area. Students can draw upon their understanding of calculating the areas of other shapes in their estimate of the areas of a circle.

Students at this level are most like to be familiar with the word 'Pi', the button on their calculator, or an approximate value for it. This example is an opportunity to revise measurement concepts but the learning intention is for students to gain a better understanding of π as a number. Ask students to research at least two interesting facts about π and then share and display this new knowledge as a class.

When discussing errors that occur when measuring physical objects, it is appropriate to consider percentage differences, rather than absolute errors. We can also take this opportunity to consolidate understanding within the Statistics strand as we are exploring the practicalities and implications of collecting data.

You can ask:

- *How accurate are the measurements taken from smaller circles? How accurate are the measurements taken from larger circles?*
- *How can you determine which of these two sets of measurements are more accurate?*

Often the data from the larger circles is more accurate as it is physically easier to collect. Even if this is not the case, if you find the errors as a percentage of the measurement being taken, the percentage error of large circles is less than the percentage error for the small circles (all other factors being the same). When investigating percentage differences in this context, we are connecting our understanding in number, statistics and measurement. In this way, we are giving students the opportunity to **work mathematically** rather than just **do mathematics**.

Check out the **Rope around the world** activity at: http://www.abc.net.au/science/surfingscientist/pdf/conundrum17.pdf

(A similar activity, 'Example 8: The race', can be found in the *Using units of measurement: Year 8* narrative.)

Proficiency: Problem Solving

Proficiency emphasis and what questions to ask to activate it in your students (Examples 8–13)

Problems are described in the AC: Mathematics as 'meaningful' or 'unfamiliar'. Students of *all* abilities and ages should be provided with experiences of both meaningful and unfamiliar problems.

Meaningful problems

Meaningful problems are set in a context that a student can project themselves into. It may be that the mathematics and strategy being applied is familiar to the student or the problem relates to their own life experience. Connecting with a context does not mean that the students have to see it as 'fun' nor does it have to relate to an immediately practical situation from daily life.

Students can connect with a problem through provocations such as the use of manipulatives (either physical or digital) or through a dramatisation (eg, a story, interesting background information, a video clip). The intention is to give students the opportunity to work as a mathematician would work, in a context that they can access at their current stage of development.

Unfamiliar problems

Fundamentally there are two groups of unfamiliar problems:

- Problems for which the students would not be able to say that they had done a similar example previously, they would therefore need to create an approach (develop a strategy).
- Problems in which the students develop a new piece of knowledge. They begin the problem by applying the knowledge/skills that they have and they complete the problem having recombined that knowledge to form a new piece of understanding.

Growth mindset: Learning that not knowing is the beginning of a learning opportunity

Unfamiliar problems tend to provoke a response of, 'I don't know', or 'I'm not sure'. Students respond differently to this feeling; some shut down, others begin to ask, 'But how could I work that out?'

In developing powerful learners we are aiming for all of our students to learn that 'not knowing' is the *beginning of a learning opportunity* and that the first move that they need to make on the journey to finding out more is to ask, 'What could I do to work this out?'

Engaging in problem solving supports the move *from tell to ask*

Instead of ***telling*** students:

- the problem to solve
- the information they'll need
- the steps they should take.

We can ***ask students to identify***:

- the problem to solve
- the information they'll need
- a possible process to use.

Proficiency: Problem Solving examples	
Example 8: Strong cordial – ratios Students solve a range of problems involving rates and ratios, with and without digital technologies.	ACMNA188 ◆
Example 9: Nana's paint mixup – ratios Students solve a range of problems involving rates and ratios, with and without digital technologies.	ACMNA188 ◆
Example 10: How do drink containers measure up? – ratios Students solve a range of problems involving rates and ratios, with and without digital technologies.	ACMNA188 ◆
Example 11: Repetitiously – recurring decimals to fractions Students investigate terminating and recurring decimals.	ACMNA184 ◆
Example 12: Percentage unchanged Students solve problems involving the use of percentages, including percentage increases and decreases, with and without digital technologies.	ACMNA187 ◆
Example 13: Coke v Sprite – percentage increase and decrease Students solve problems involving the use of percentages, including percentage increases and decreases, with and without digital technologies. Students solve a range of problems involving rates and ratios, with and without digital technologies.	ACMNA187 ◆ ACMNA188 ◆

Example 8: Strong cordial – ratios

ACMNA188 ◆
Students solve a range of problems involving rates and ratios, with and without digital technologies.

Questions from the BitL tool
Problem solving proficiency:
Interpret; Model and plan; Solve and check; Reflect.
Reasoning proficiency:
What can you infer?

Instead of ***telling*** students about ratios, we can challenge students to explore the concepts for themselves, by ***asking*** questions.

There is an opportunity in this activity to discuss the difference between **part:part** and **part:whole** comparisons. The cordial in glass A contains one quarter syrup when the ratio of syrup to water is 1:3 (see Figure 4). As this often causes confusion for students, take the opportunity to describe these mixtures both ways.

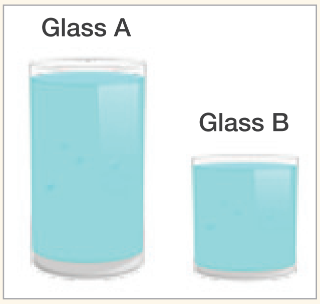


Figure 4

Use two containers, one twice the volume of the other. Mix up cordial to different strengths in the following way:

In glass A: $\frac{1}{4}$ cordial syrup $\frac{3}{4}$ water
In glass B: $\frac{1}{3}$ cordial syrup $\frac{2}{3}$ water

The manufacturers recommend that the cordial should be no more than 30% syrup. Ask students:

- *If it is decided that both glasses are tipped into a jug to change the strength of the cordial, what would be the resulting strength of the mixture?*

Students may solve this problem using a convenient specific example and draw it to scale. A more generalised argument is to consider the combined mixture (see Figure 5):

The **large glass (glass A)** makes up two thirds of the combined mixture, and $\frac{1}{4}$ of this is cordial, ie $\frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$. So $\frac{1}{6}$ of the combined mixture is cordial from the large glass.

The **small glass (glass B)** makes up one third of the combined mixture, and $\frac{1}{3}$ of this is cordial, ie $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$. So $\frac{1}{9}$ of the combined mixture is cordial from the small glass.

In total: $\frac{1}{6} \times \frac{1}{9} = \frac{5}{18} \approx 27.8\% < 30\%$

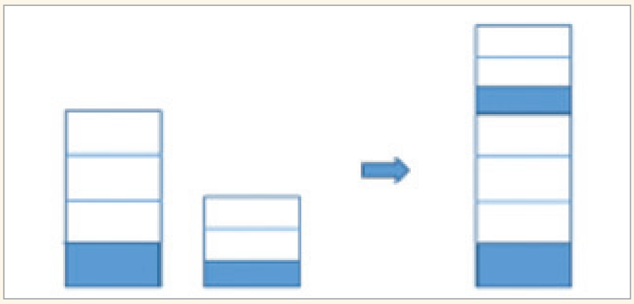


Figure 5

Interpret

What are you trying to find? What's an answer that's too big? What's an answer that's too small? What's a bit closer to the answer that you think you'll find? What do you need to show to answer that question? What information is helpful? What information is not helpful? (Remember that telling students or prompting heavily is stealing the opportunity for them to realise for themselves later on in the task.)

Model and plan

Do you have an idea? How might you start? What equipment will be helpful? Would it help to draw a diagram? (Ask students to speak to someone who they think is being a good problem solver today and ask them to show them what they are trying.)

Solve and check

Question to be used only after students have grappled with the problem for a few minutes: What makes the use of fractions difficult in this problem? (The whole is not the same for both glasses. A quarter of the big glass is not the same amount as a quarter of the smaller one.)

How might you consider the amounts so that a comparison can be made? Does that seem right to you? Do other people think that too? (Students can explore these specific quantities (1 litre and 2 litre) but then should convince themselves that it will work for any quantity. Scale diagrams can be used to give approximate solutions.)

Reflect

Pair up with someone who did it differently. How do your methods compare? What do you like about each other's strategy? How could you help each other to improve? Have you reached the same/a similar conclusion? How efficient was your strategy? Is there something that you would do differently next time? Is there a more efficient way?

The **Mixing lemonade** digital activity from the NRICH website can also support problem solving in this context. **The link to the problem on the NRICH website is: http://nrich.maths.org/6870**

Example 9: Nana's paint mixup – ratios

ACMNA188 ◆
Students solve a range of problems involving rates and ratios, with and without digital technologies.

Questions from the BitL tool
Problem solving proficiency:
Interpret; Model and plan; Solve and check; Reflect.
Reasoning proficiency:
What can you infer?

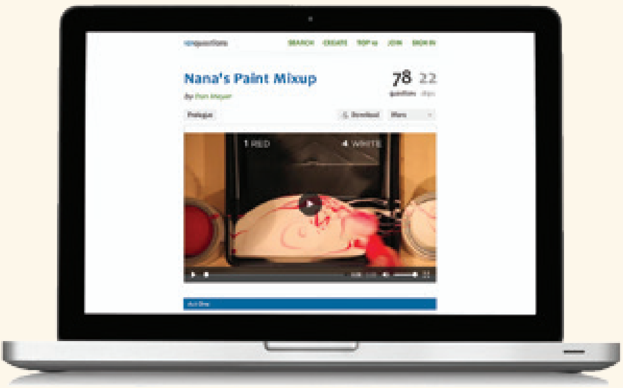
Instead of ***telling*** students about ratios, we can challenge students to explore the concepts for themselves, by ***asking*** questions.

This activity is a ***Three-Act Maths Task*** from Dan Meyer's blog: 101 questions. It should be presented to students along with the question:

Is it possible to fix the mixup?

The video activity sets a challenge for students to correct an error made in mixing paint in a given ratio.

The **Nana's paint mixup** activity can be accessed at: http://www.101qs.com/2841-nanas-paint-mixup



Example 10: How do drink containers measure up? – ratios

ACMNA188 ◆
Students solve a range of problems involving rates and ratios, with and without digital technologies.

Questions from the BitL tool
Problem solving proficiency:
Interpret; Model and plan; Solve and check; Reflect.
Reasoning proficiency:
What can you infer?

Instead of ***telling*** students about ratios, we can challenge students to explore the concepts for themselves, by ***asking*** questions.

Present a range of drink bottles to students and ask them to estimate their relative capacity (like those in Figure 6).

Figure 6

Initially, ask the students to estimate their ranking from smallest to largest, then ask:

- *What makes it hard to rank them with confidence? How might you test your ranking?* (Some containers have outer coverings or lids which make them taller, some are taller but thinner and some do not have a constant width.)
- *Can you estimate their relative capacities as accurately as possible?* (Students might compare all others to the smallest (or the biggest) container or others might do more relational comparisons, such as, that one is about one tenth bigger, the next is another about a tenth again, the third one is midway between the largest and the smallest, etc.)

Ask students to record their estimates as fully and accurately as they can.

Prediction supports the development of conceptual understanding

When asked to make predictions, students are challenged to think more deeply about the process they are about to observe. Asking what the difference 'might be' rather than what it 'will be' invites thinking rather than 'seeking the correct answer'. Students are often motivated to experiment and take notice of the results to see how well the results match their predictions, but we can support a growth mindset through valuing their thinking rather than the accuracy of their predictions.

Using these estimates, ask the students to make comparisons that do not involve any fractions: 11 of the larger one would be the same as 10 of the smaller one. Ask students:

- *Can you convince me that the two comparisons you have made would be the same? Which of the comparisons is easier to test? Why?* (Allow students to test their estimates, while ensuring they conserve the water.)

'Squirt' is a digital activity from the Scootle website which allows students to practise their estimations and ratios (and conserve water).

The **Squirt** activity can be found at: http://www.scootle.edu.au/ec/viewing/L1996/index.html

Students engage with digital activities in a far more meaningful way if they have already experienced the concrete materials.

Example 11: Repetitiously – recurring decimals to fractions

ACMNA184 ◆
Students investigate terminating and recurring decimals.

Questions from the BitL tool

Problem solving proficiency:
Interpret; Model and plan; Solve and check; Reflect.

Reasoning proficiency:
What can you infer?

Instead of ***telling*** students about ratios, we can challenge students to explore the concepts for themselves, by ***asking*** questions.

This activity is from the NRICH website.

Begin by asking students:

- *What is the connection between these pairs of numbers? What do you notice? What do you wonder?*
  0.33333333 … and 3.3333333 …
  0.77777777 … and 7.7777777 …

Comment on this student's thinking:

I was thinking about 0.333333 … that goes on forever. Ten times that would be 3.333333 … If you subtracted them you would just end up with three. So then I did this:

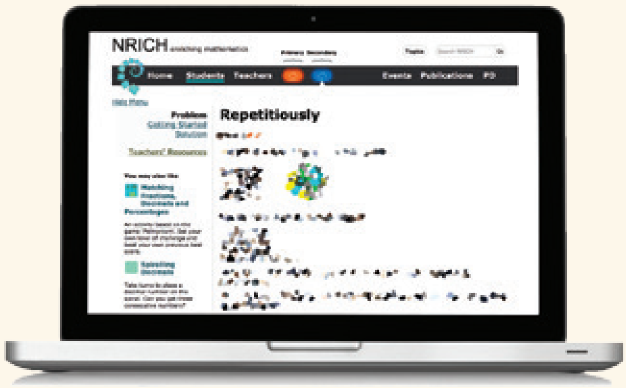
$10x - x = 3$

$9x = 3$

$x = \frac{3}{9} = \frac{1}{3}$

I reckon that works, so now I am going to try to do the same sort of thing for other decimals, even ones I don't know like 0.232323 … or 0.567567 … What do you think?

When considering fractions with 2 digits recurring, consider 100*x*, 3 digits recurring, 1000*x*, etc.



The link to the problem on the NRICH website is: http://nrich.maths.org/1853

Example 12: Percentage unchanged

ACMNA187 ◆
Students solve problems involving the use of percentages, including percentage increases and decreases, with and without digital technologies.

Questions from the BitL tool

Problem solving proficiency:
Interpret; Model and plan; Solve and check; Reflect.

Reasoning proficiency:
What can you infer?

Instead of ***telling*** students about ratios, we can challenge students to explore the concepts for themselves, by ***asking*** questions.

This activity is from the NRICH website.

How does increasing one dimension of a shape affect the area? This task from NRICH explores percentage rather than absolute change in one dimension of a rectangle. At this level, students could explore the effect of a percentage change in dimensions for other shapes such as parallelograms, trapeziums, rhombuses and kites.

The link to the problem on the NRICH website is: https://nrich.maths.org/515

Example 13: Coke v Sprite – percentage increase and decrease

ACMNA187 ◆

Students solve problems involving the use of percentages, including percentage increases and decreases, with and without digital technologies.

ACMNA188 ◆

Students solve a range of problems involving rates and ratios, with and without digital technologies.

Questions from the BitL tool

Problem solving proficiency:

Interpret; Model and plan; Solve and check; Reflect.

Reasoning proficiency:

What can you infer?

Instead of ***telling*** students about ratios, we can challenge students to explore the concepts for themselves, by ***asking*** questions.

This activity is a ***Three-Act Maths Task*** from Dan Meyer. It should be presented to students along with the question:

Which glass contains most of the original soda?

A short video shows small exchanges of liquid from one glass to another, posing the question, '*Now which glass contains most of the original soda?*'. Students work with ratios and mixtures to discuss their ideas.

The **Coke v Sprite** activity can be accessed at: http://mrmeyer.com/threeacts/cokevsprite/

Connections between 'Real numbers' and other maths content

There are many opportunities to connect to other content in the AC: Mathematics, when we use Real numbers as a starting point.

Here are just some of the possible connections that can be made:

Mathematics: Year 8	
Whilst working with Real numbers, connections can be made to:	**How the connection might be made:**
Students use index notation with numbers to establish the index laws with positive integral indices and the zero index. ACMNA182	Refer to: **Example 3: What number times by itself gives …?**
Students simplify algebraic expressions involving the four operations. ACMNA192	Refer to: **Example 11: Repetitiously – recurring decimals to fractions**
Students find perimeters and areas of parallelograms, trapeziums, rhombuses and kites. ACMMG196	Refer to: **Example 12: Percentage unchanged**
Students plot linear relationships on the Cartesian plane with and without the use of digital technologies. ACMNA193	Refer to: **Example 6: Hardware ratios** **Example 7: Calculating Pi**
Students investigate the relationship between features of circles such as circumference, area, radius and diameter and use formulas to solve problems involving circumference and area. ACMMG197	Refer to: **Example 7: Calculating Pi** **Example 12: Percentage unchanged**
Students explore the practicalities and implications of obtaining data through sampling using a variety of investigative processes. ACMSP206	Refer to: **Example 7: Calculating Pi**

Making connections to other learning areas

We know that when our students meet a concept frequently and in different contexts, they have a greater chance of developing understanding. With this in mind, it is our responsibility to help our students to make these connections by intentionally designing tasks that connect a number of different content descriptions. Alternatively, connections can be made through questioning individual or small groups of students.

‘Real numbers’ from Year 7 to Year 10A

The AC: Mathematics year level content descriptions shown here have been colour coded to highlight the following curriculum aspects of working with Real numbers:

Recognise, describe and represent real numbers ◆
In Foundation to Year 3 students recognise, describe and represent fractions and decimals. In Year 10A students are mostly recognising and describing more abstract real numbers.

Compare and order real numbers ◆
In Year 4 to Year 5 students are also expected to compare and order fractions and decimals.

Convert and calculate using real numbers ◆
In Year 6 students mostly convert and calculate using fractions and decimals. In Year 7 students mostly convert and calculate using fractions, decimals and percentages.

Apply and solve problems using real numbers ◆
In Years 8 to Year 9 students are mostly solving problems using percentages, rates and ratios.

Year level	‘Fractions and decimals’ content descriptions from the AC: Mathematics: Year 1 to Year 6
Year 1 ◆	Students recognise and describe one-half as one of two equal parts of a whole. ACMNA016
Year 2 ◆	Students recognise and interpret common uses of halves, quarters and eighths of shapes and collections. ACMNA033
Year 3 ◆	Students model and represent unit fractions including 1/2, 1/4, 1/3, 1/5 and their multiples to a complete whole. ACMNA058
Year 4 ◆	Students investigate equivalent fractions used in contexts. ACMNA077
Year 4 ◆ ◆ ◆	Students count by quarters halves and thirds, including with mixed numerals. Locate and represent these fractions on a number line. ACMNA078
Year 4 ◆ ◆	Students recognise that the place value system can be extended to tenths and hundredths. Make connections between fractions and decimal notation. ACMNA079
Year 5 ◆ ◆	Students compare and order common unit fractions and locate and represent them on a number line. ACMNA102
Year 5 ◆ ◆	Students investigate strategies to solve problems involving addition and subtraction of fractions with the same denominator. ACMNA103
Year 5 ◆	Students recognise that the place value system can be extended beyond hundredths. ACMNA104
Year 5 ◆ ◆	Students compare, order and represent decimals. ACMNA105
Year 6 ◆ ◆	Students compare fractions with related denominators and locate and represent them on a number line. ACMNA125
Year 6 ◆ ◆	Students solve problems involving addition and subtraction of fractions with the same or related denominators. ACMNA126
Year 6 ◆	Students find a simple fraction of a quantity where the result is a whole number, with and without digital technologies. ACMNA127
Year 6 ◆	Students add and subtract decimals, with and without digital technologies, and use estimation and rounding to check the reasonableness of answers. ACMNA128
Year 6 ◆	Students multiply decimals by whole numbers and perform divisions by non-zero whole numbers where the results are terminating decimals, with and without digital technologies. ACMNA129
Year 6 ◆	Students multiply and divide decimals by powers of 10. ACMNA130
Year 6 ◆	Students make connections between equivalent fractions, decimals and percentages. ACMNA131

Year level	'Real numbers' content descriptions from the AC: Mathematics: Year 7 to Year 10A
Year 7 ◆ ◆	Students compare fractions using equivalence. Locate and represent positive and negative fractions and mixed numbers on a number line. ACMNA152
Year 7 ◆ ◆	Students solve problems involving addition and subtraction of fractions, including those with unrelated denominators. ACMNA153
Year 7 ◆	Students multiply and divide fractions and decimals using efficient written strategies and digital technologies. ACMNA154
Year 7 ◆	Students express one quantity as a fraction of another, with and without the use of digital technologies. ACMNA155
Year 7 ◆	Students round decimals to a specified number of decimal places. ACMNA156
Year 7 ◆	Students connect fractions, decimals and percentages and carry out simple conversions. ACMNA157
Year 7 ◆	Students find percentages of quantities and express one quantity as a percentage of another, with and without digital technologies. ACMNA158
Year 7 ◆ ◆	Students recognise and solve problems involving simple ratios. ACMNA173
Year 8 ◆	Students investigate terminating and recurring decimals. ACMNA184
Year 8 ◆	Students investigate the concept of irrational numbers, including π. ACMNA186
Year 8 ◆	Students solve problems involving the use of percentages, including percentage increases and decreases, with and without digital technologies. ACMNA187
Year 8 ◆	Students solve a range of problems involving rates and ratios, with and without digital technologies. ACMNA188
Year 9 ◆	Students solve problems involving direct proportion. Explore the relationship between graphs and equations corresponding to simple rate problems. ACMNA208
Year 9 ◆	Students apply index laws to numerical expressions with integer indices. ACMNA209
Year 9 ◆	Students express numbers in scientific notation. ACMNA210
Year 10A ◆ ◆	Students define rational and irrational numbers and perform operations with surds and fractional indices. ACMNA264
Year 10A ◆ ◆ ◆	Students use the definition of a logarithm to establish and apply the laws of logarithms. ACMNA265

Numeracy continuum: Using fractions, decimal, percentages, ratios and rates	
End Foundation	Recognise that a whole object can be divided into equal parts. Identify quantities such as more, less and the same in everyday comparisons.
End Year 2	Visualise and describe halves and quarters. Solve problems using halves and quarters.
End Year 4	Visualise, describe and order tenths, hundredths, 1-place and 2-place decimals. Solve problems using equivalent fractions for tenths, hundredths, 1-place and 2-place decimals.
End Year 6	Visualise, describe and order equivalent fractions, decimals and simple percentages. Solve problems using equivalent fractions, decimals and simple percentage.
End Year 8	Visualise and describe the proportions of percentages, ratios and rates. Solve problems using simple percentages, ratios and rates.
End Year 10	Illustrate and order relationships for fractions, decimals, percentages, ratios and rates. Solve problems involving fractions, decimals, percentages, ratios and rates.

Source: ACARA, Australian Curriculum: Mathematics

Resources

NRICH website

http://nrich.maths.org

In this conceptual narrative we have highlighted the possibility of using tasks from an organisation called NRICH enriching mathematics.

The NRICH website contains a large collection of high quality maths problem solving tasks, together with suggestions about content that may be related to the task, ways to get started and different (valid) solutions that have been submitted by students from around the world.

Dan Meyer's blog: 101 questions

http://www.101qs.com

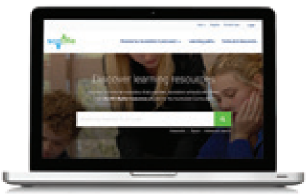
Dan's blog contains images and short films that can be presented to students along with the question: *What's the first question that comes to mind?*

A spreadsheet of ***Dan Meyer's Three-Act Maths Tasks*** can be accessed at http://bit.ly/DM3ActMathTasks.

Scootle

https://www.scootle.edu.au/ec/p/home

This website has over 20,000 quality-assured digital learning resources aligned to the Australian Curriculum. You can filter your search to uncover a wealth of relevant teaching and learning items.



Estimation 180

http://www.estimation180.com

Estimation 180 is a website with a bank of daily estimation challenges to help students to improve both their number sense and problem solving skills.

reSolve: maths by inquiry

https://www.resolve.edu.au

This website provides classroom resources for years F to 10 that promote fluency, deep understanding, strategic problem solving, and mathematical reasoning. Each classroom resource is designed to develop progressive understanding through tasks that encourage a spirit of inquiry.

Plus Magazine

https://plus.maths.org

An online magazine which aims to introduce readers to the beauty and the practical applications of mathematics. It includes articles, teaching packages, puzzles and more. It has examples of mathematical modelling from scaffolded to open in authentic contexts.

Do you want to feel more confident about the maths you are teaching?

Do you want activities that support you to embed the proficiencies?

Do you want your students thinking mathematically rather than just doing maths?

If you answered **yes** to any of these questions, then this resource is for you.

Packed full of examples, along with questions you can ask students as they engage in their learning, this resource supports you to develop confidence in teaching the Australian Curriculum: Mathematics.

Produced by Learning Improvement